Sizes of Linear Descriptions in Combinatorial Optimization

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Zusammenfassung

Ein zentrales Paradigma der Kombinatorischen Optimierung ist, zulässige Objekte eines diskreten Optimierungsproblems mit euklidischen Vektoren zu identifizieren, so dass sich gegebene Zielfunktionen in lineare Funktionen über diesen Punkten übersetzen lassen. Bei diesem Ansatz beschreibt man die Menge X dieser Vektoren, die in der Regel ganzzahlige Einträge haben, implizit als Menge aller ganzzahligen Punkte in einem Polyeder P. Da sich Polyeder als (affine Projektionen von) Lösungsmengen linearer Ungleichungssysteme beschreiben lassen, ermöglicht dies die Anwendbarkeit von allgemeingültigen Algorithmen der Linearen Ganzzahligen Optimierung, die oft effiziente Lösungsverfahren in Theorie und Praxis liefern. Dabei spielt die Anzahl der Ungleichungen dieser Systeme oft eine entscheidende Rolle. In dieser Arbeit widmen wir uns theoretischen Fragestellungen bezüglich der Größe zweier prominenter Varianten solcher Beschreibungen. Dabei erarbeiten wir neue obere Schranken (Konstruktionen) als auch untere Schranken (Obstruktionen) und stellen diese in Zusammenhang mit existierenden Resultaten.

Im ersten Kapitel dieser Arbeit beschäftigen wir uns mit dem Konzept der *erweiterten Formulierungen*. In diesem Fall schränkt man die Wahl von P auf die konvexe Hülle von X ein, beschreibt P aber als affine Projektion eines beliebigen Polyeders Q. Wir betrachten in diesem Teil zunächst unterschiedliche Techniken zur Konstruktion von erweiterten Formulierungen. Dabei zeigen wir unter anderem, dass Unabhängigkeitspolytope großer Klassen von Matroiden durch erweiterte Formulierungen beschrieben werden können, deren Größe nur polynomiell von der Dimension der Polytope abhängt. Zu diesen Klassen gehört beispielsweise die Menge der regulären Matroide.

Des Weiteren beschäftigen wir uns mit Beweisstrategien zur Gewinnung unterer Schranken an Größen erweiterter Formulierungen. Wir gehen beispielsweise auf Anwendbarkeit und Limitierungen einiger bekannter Techniken mit Hinblick auf relevante Fragestellungen, wie etwa der Fragen nach Erweiterungskomplexitäten von Spannbaumpolytopen und Unabhängigkeitspolytopen allgemeiner Matroide, ein. Als ein weiteres Hauptresultat dieses Teils liefern wir einen stark vereinfachten Beweis für den Fakt, dass die Anzahl der Ungleichungen

einer jeden erweiterten Formulierung des Korrelationspolytops superpolynomiell in dessen Dimension wächst. Unser Resultat stellt die aktuell beste untere Schranke an diese Größe dar.

Das zweite Kapitel dieser Arbeit widmen wir dem Konzept der linearen *Relaxierungen*. Dabei wird P nicht als Projektion, sondern im Originalraum beschrieben, ist aber ein beliebiges Polyeder, dessen ganzzahlige Punkte mit denen der Menge X übereinstimmen. Wir führen den Begriff der Relaxierungskomplexität von X ein, die als kleinste Anzahl von Facetten eines solchen Polyeders P definiert ist und die zentrale Größe dieses Kapitels darstellt. Wir leiten zunächst obere Schranken an die Relaxierungskomplexität für ausgewählte sowie allgemeine Mengen X her. Zudem geben wir einen Algorithmus an, der die Relaxierungskomplexität (sowie eine kleinstmögliche Relaxierung) einer zweidimensionalen Menge X berechnet.

Im zweiten Teil dieses Kapitels erarbeiten wir untere Schranken an die Relaxierungskomplexität. Dabei konzentrieren wir uns zunächst auf konkrete einfache Mengen X wie beispielsweise die Eckenmenge des 0/1-Hyperwürfels, geben aber auch eine untere Schranke an den Erwartungswert der Relaxierungskomplexität einer zufälligen Teilmenge dieser Menge. Anschließend entwickeln wir eine allgemeine Methode, um untere Schranken an die Relaxierungskomplexität zu erhalten. Diese wenden wir auf zahlreiche Mengen an, die mit klassischen Optimierungsproblemen assoziiert sind. Unter anderem zeigen wir, dass jede Relaxierung der Menge der charakteristischen Vektoren von Hamilton-Kreisen des vollständigen Graphen (die die zulässigen Objekte des Problems des Handlungsreisenden auf kanonische Art beschreibt), exponentiell groß in der Anzahl der Knoten sein muss.

Summary

A central paradigm in combinatorial optimization is to identify feasible objects of discrete optimization problems with Euclidean vectors such that given objective functions translate into linear functions over these points. In this approach, the set X of these vectors, whose entries are usually integers, is implicitly described as the set of integer points in some polyhedron P. As polyhedra are (affine projections of) solution sets of systems of linear inequalities, this enables the applicability of general-purpose algorithms from the field of linear integer programming, which often yield theoretically and practically efficient solution methods. Here, the number of inequalities in such systems often plays a crucial role. This work addresses questions regarding the sizes of two prominent variants of such descriptions. We provide new theoretic results referring to both upper bounds (*constructions*) and lower bounds (*obstructions*), and place them in context with existing results.

The first chapter of this thesis is concerned with the concept of *extended formulations*. In this case, P is restricted to coincide with the convex hull of X but is described as an affine projection of some polyhedron Q. We first consider different techniques for deriving extended formulations. As one application, we show that independence polytopes of large classes of matroids can be described by extended formulations whose sizes can be bounded polynomially in the dimensions of the polytopes. These classes include, for instance, the set of regular matroids.

Furthermore, we address proof strategies for establishing lower bounds on sizes of extended formulations. We consider the applicability and limitations of several known techniques with respect to relevant open questions, such as questions referring to the extension complexities of spanning-tree polytopes and independence polytopes of general matroids. Another result contained in this part is a simplified proof of the fact that the number of inequalities in any extended formulation of the correlation polytope grows superpolynomially in its dimension. Our result establishes the currently best known lower bound on this size.

The second chapter of this thesis is concerned with the concept of linear *relaxations*. In this case, P is described in its ambient space

(instead via projection) but can be any polyhedron whose set of integer points coincides with X. We introduce the relaxation complexity of a set X, which is defined as the smallest number of facets of any such polyhedron and will be the quantity of main interest of this chapter. We start by deriving upper bounds on the relaxation complexities of specific choices for X as well as for general sets X. In addition, we give an algorithm computing the relaxation complexity (as well as a minimum-size relaxation) of any two-dimensional set X.

The second part of this chapter is concerned with the development of lower bounds on the relaxation complexity. We first focus on specific simple sets X such as the vertex set of the 0/1-hypercube, but also give a lower bound on the expected value of the relaxation complexity of a random subset of this set. Afterwards, we develop a general technique to establish lower bounds on the relaxation complexity. We apply this method to several sets that are associated to classical optimization problems. For instance, we show that every relaxation of the set of characteristic vectors of Hamilton-cycles of the complete graph (which canonically describe the feasible objects of the traveling-salesman problem) has exponential size in the number of nodes.

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[1] "Extended Formulations in Combinatorial Optimization", KA 1616/4-1

[2] a two-month research stay at the University of Tartu, Estonia, and a congress trip to the ISMP 2015, Pittsburgh, USA

I strongly believe that experiencing so many great opportunities is a result of the unlimited encouragement that I received from my parents Klaus and Katrin, who always put me and my brother Thomas first. I am deeply grateful to all three of them for always believing in me and supporting me in each phase of my life.

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This thesis is dedicated to all my family.

Contents

1

Introduction

Classical combinatorial optimization asks for finding an optimum object in a finite set of objects. Typically, the collection of feasible solutions is a set of certain subsets of a finite ground set E, where each objective function can be represented by a vector $c \in \mathbb{R}^E$ such that the value of a feasible set F is equal to $\sum_{e \in F} c_e$. Here, usually E and c are given explicitly as an input, whereas the set of feasible solutions is defined only implicitly. A central paradigm in combinatorial optimization is to identify each feasible subset F with its *characteristic vector* $\chi(F) \in \{0,1\}^E$ with $\chi(F)_e = 1 \iff e \in F$ and to consider the (equivalent) problem of solving

$$\max / \min \{\langle c, x \rangle \mid x \in X\}, \tag{1.1}$$

where we define

$$X := \{\chi(F) : F \subseteq E \text{ feasible}\}.$$

This very basic idea transfers optimization tasks into problems embedded in some Euclidean space, a mathematical structure, which, for various computational tasks, provides a rich theory of structural and algorithmic results as well as highly sophisticated software implementations that have been developed over many decades. Clearly, as the set X usually contains an enormous number of feasible points, the applicability (and performance) of algorithms solving (1.1) heavily depends on the actual representation of X.

Since $x \mapsto \langle c, x\rangle$ is a linear function, in the description of problem (1.1) we clearly can replace X by its convex hull. Thus, it is natural to model this problem as a *linear program* of type

$$\max / \min \left\{ \langle c, x\rangle \mid x \in \mathbb{R}^E,\ Ax \leq b \right\} \tag{1.2}$$

for some system of linear inequalities $Ax \leq b$ being an outer description of $\operatorname{conv}(X)$. This classical approach, which has been pioneered by Edmonds and lead to many fundamental achievements in combinatorial optimization, has an important advantage: Once the systems $Ax \leq b$ can be constructed in time polynomial in $|E|$, problem (1.1) can be solved in time polynomial in $|E|$ and the encoding length of c. Clearly, in this case the polytope $\operatorname{conv}(X)$ is restricted to have only polynomially many facets, which is not the case for most (even polynomial-time tractable) problems.

A well-known result of Grötschel, Lovász and Schrijver [39] states that – in order to derive a polynomial-time algorithm for solving (1.1) – the systems $Ax \leq b$ actually do not need to have small size but are only required to yield a polynomial-time algorithm to solve a certain separation problem. However, using such a description amounts to implementing separation routines rather than using linear programming solvers in a black-box way. In some sense, this is incompatible with an important aspect of the success of many mathematical programming frameworks, namely the ability to *easily* formulate problems and use existing algorithms and software – even as a non-mathematician. Therefore, in this work we restrict our attention to formulations that have *small size* and hence potentially can be easily written down *explicitly*. Here, "small" is always understood relatively to the cardinality of E and usually means "polynomial".

Note that the size of any linear programming formulation as in (1.2) is already determined by X (up to redundancy). On the other hand, by considering more general models, it is possible to come up with significantly smaller formulations. In this thesis, we will focus on two prominent types of such modeling approaches.

Clearly, there is no need to restrict a general linear program solving (1.1) to be formulated in the ambient space of X. Instead, it is common to make use of *additional variables* $y = (y_1, \dots, y_k)$ and to find systems of linear inequalities $Ax + By \leq b$ such that

$$\max / \min \left\{ \langle c, x \rangle \mid x \in \mathbb{R}^E,\ y \in \mathbb{R}^k,\ Ax + By \leq b \right\}$$

yields the correct value for all objectives $c \in \mathbb{R}^E$. Linear programs of that type are called *extended formulations* and have already been used in the early days of combinatorial optimization. It is easy to come up with sets X for which any linear program that is formulated only in the x-variables has exponentially many inequalities but for which there exist extended formulations of only polynomial size. Geometrically, extended formulations correspond to polyhedra (here defined by $\{(x, y) \mid Ax + By \leq b\}$) that can be projected onto $\mathrm{conv}(X)$ by some affine map. The concept of describing polytopes in such a way is the subject of the first chapter of this thesis.

The investigation of extended formulations is a very active research field, which has received a renewed attention in the last few years. On one side, there is a constant interest in finding constructions of small-size extended formulations for relevant problems. On the other side, as polynomial-time constructable extended formulations describe optimization problems that can be solved by polynomial-time algorithms, one expects that problems that are assumed to be hard do not admit small-size formulations. In fact, recent developments established a series of results concerning the limitations of this concept. In particular, they yield many examples of combinatorial-optimization problems that indeed cannot be modeled by polynomial-size extended formulations. Moreover, this has turned out to be true even for some polynomial-time solvable problems.

This situation changes drastically if one allows to impose additional *integrality constraints* on subsets of variables. For every such formulation, as the set X only consists of integer points, we can clearly require all x-variables to be integer without changing its correctness (and size). Thus, in its simplest form, a formulation of this type reads as follows:

$$\max / \min \left\{ \langle c, x \rangle \mid x \in \mathbb{Z}^E,\ y \in \mathbb{R}^k,\ Ax + By \leq b \right\} \tag{1.3}$$

It turns out that this model is already powerful enough to describe all (in a certain sense) reasonable combinatorial optimization problems by polynomial-size systems $Ax + By \leq b$. Such formulations are called *mixed-integer programs* and are extensively used in order to formulate most practical combinatorial optimization problems. Unfortunately, mixed-integer programs do not seem to be theoretically tractable. Nevertheless, they are closely related to linear programs, which allows modern software to often solve these problems in practically efficient time.

Geometrically, formulations of type (1.3) correspond to polyhedra that can be projected down to some polyhedron $P \subseteq \mathbb{R}^E$ whose set of integer points coincides with X. Interestingly, for many sets X associated to prominent (and often hard) combinatorial optimization problems the polyhedron P can already be chosen to have polynomially many facets. In this case, problem (1.1) can be rewritten as an integer program

$$\max / \min \left\{ \langle c, x \rangle \mid x \in \mathbb{Z}^E,\ Ax \leq b \right\} \tag{1.4}$$

for some polynomial-size system $Ax \leq b$, which does not use additional variables. Throughout this work, any polyhedron $P \subseteq \mathbb{R}^E$ with $P \cap \mathbb{Z}^E = X$ will be called a *relaxation* for X. Describing sets X by relaxations will be the subject of the second chapter of this thesis.

In contrast to extended formulations, the constructability of small-size relaxations does not imply efficient algorithms for optimizing linear functions over X. In this sense, there are no indications which combinatorial optimization problems cannot be modeled with polynomial-size integer programs without additional variables. In particular, although the concept is potentially more powerful, there seems to be no obvious reason why (mixed) integer programs with additional variables can yield provably smaller formulations than integer programs without additional variables.

Contribution

As already mentioned, every reasonable[1] combinatorial optimization problem that ask for optimizing a linear function over some finite set X can be formulated as polynomial-size mixed-integer programs by using the concepts of lifting *and* relaxing.

[1] see Section 7.1 for a precise statement

In this thesis, we investigate how the situation changes if one drops one of these two paradigms. To this end, for each restricted model we provide new theoretic results referring to both upper bounds (*constructions*) and lower bounds (*obstructions*) on sizes of formulations, and place them in context with existing results.

The main motivation for this work is two-fold: First, we hope that our work contributes to a better understanding of general possibilities and limits of formulating combinatorial optimization problems as linear or mixed-integer programs. In particular, it shows that projection, i.e., the use of additional variables, is a *provably* powerful technique that allows for significantly smaller descriptions in many models. Second, we aim to convince the reader that the question for small models of certain types lead to interesting mathematical questions and proofs combining results from different areas including polyhedral combinatorics, convex geometry and matroid theory. A summary of all explicit results of our work is given in the respective beginnings of the two chapters of this thesis.

Preliminaries Throughout this work, we assume familiarity with basic facts about convex geometry and in particular polyhedra. For detailed background information we refer to the books of Schrijver [73] and Ziegler [81]. Furthermore, most questions addressed here are related to classical problems in combinatorial optimization. Thus, the books of Korte & Vygen [52] and Schrijver [74] might serve as additional important references. Most terminology and notational conventions we use are adapted from standard literature and can be found in the mentioned references.

Part I

Describing Polytopes

2

Describing Polytopes: Definitions & Background

Most polytopes that are associated to classical combinatorial-optimization problems have many facets compared to their dimension, making it impossible to exactly describe them by a small number of linear inequalities in their ambient space. In contrast, there are many examples of polytopes P that can be written as affine projections of other polyhedra having much less facets than P. The theory of extended formulations deals with the concept of describing polytopes in such a way, and will be the subject of this chapter.

Given polyhedra $P \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^q$ such that there exists an affine map $\pi\colon \mathbb{R}^q \to \mathbb{R}^p$ with $\pi(Q) = P$, we say that (Q, π) is an *extension* of P. The term *extended formulation* usually refers to an outer description of Q by means of linear inequalities and equations. A central property of an extension (Q, π) is clearly its *size*, which is defined as the number of facets of Q. For each polyhedron P, its *extension complexity* is then defined as the smallest size of any of its extensions and denoted by $\mathrm{xc}(P)$. Equivalently, the extension complexity of P is the smallest number of inequalities in any extended formulation for P. It is easy to see that, if P is a polytope, every extended formulation can be manipulated in a way such that the number of inequalities dominates the number of equations and the number of variables.

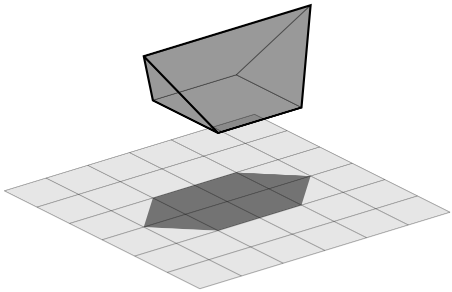

Figure 2.1: A hexagon (shadow) as a projection of a triangular prism.

Note that in our introduction we assumed the map π to be an orthogonal coordinate projection. However, this can be easily achieved by a change of the affine basis, which corresponds to an affine transformation of Q that does not increase the number of its facets.

As a simple illustration, Figure 2.1 shows a hexagon having six facets that can be represented as an orthogonal projection of a three-dimensional polytope having only five facets. Another very basic example is provided by the following observation: Consider any polytope $P \subseteq \mathbb{R}^p$ being the convex hull of some points $p_1, \dots, p_q \in \mathbb{R}^p$. Setting $Q := \left\{\lambda \in \mathbb{R}^q \mid \lambda \geq \mathbb{O}, \sum_{i=1}^{q} \lambda_i = 1\right\}$ and defining $\pi \colon \mathbb{R}^q \to \mathbb{R}^p$ via $\pi(\lambda) = \sum_{i=1}^{q} \lambda_i p_i$, we clearly have that Q is an extension of P. Since the number of facets of Q is only q, we obtain that the extension complexity of P is at most the number of its vertices. Thus, for polytopes P having much less vertices than facets (e.g., cross polytopes or cyclic polytopes), we already have a large gap between $\mathrm{xc}(P)$ and the number of facets of P.

However, there are also polytopes having both exponentially many facets and vertices but whose extension complexity can be bounded by a polynomial (in their dimension). One of the prime examples of such polytopes is the *spanning-tree polytope* $P_{\text{sp.trees}}(G)$ of a connected undirected graph $G = (V, E)$, which is defined as the convex hull of characteristic vectors of spanning trees in G. A *spanning tree* is a subset of edges $F \subseteq E$ that contains no cycle and forms a connected subgraph on all nodes of V. It is well known that, for general graphs, both the number of facets and the number of vertices of $P_{\text{sp.trees}}(G)$ grow exponentially. In contrast, as we will see in Section 3.2, the spanning-tree polytope can be described by compact extended formulations using only $O(|V| \cdot |E|)$ many inequalities.

In fact, there are many polytopes associated to other combinatorial optimization problems that can also be described by polynomial-size extended formulations (but having exponentially many facets and vertices). For a collection of many such examples, we refer to the survey of Conforti, Cornuéjols & Zambelli [17]. Not surprisingly, for all these (families of) polytopes there exist polynomial-time algorithms for optimizing linear functions over them – simply because they correspond to classical combinatorial problems that are polynomial-time tractable. Conversely, one would not expect that polytopes associated to computationally hard problems have small extension complexity.

In his seminal paper [80], Yannakakis addressed this issue and raised the question for superpolynomial lower bounds on the extension complexity of the *traveling-salesman polytope*. This polytope, which we will denote here by $P_{\text{TSP}}(n)$, is defined as the convex hull of characteristic vectors of Hamiltonian cycles in the complete undirected graph on n nodes, is associated to the traveling-salesman problem, which is well-known to be $\mathcal{NP}$-hard. A *Hamiltonian cycle* is a collection of edges that forms a single cycle visiting each node exactly once. In the same paper, the same question was asked for the *matching polytope* $P_{\text{match}}(n)$, which is defined as the convex hull of characteristic vectors of matchings in the complete undirected graph on n nodes. A *matching* is a collection of edges such that no two edges intersect in a common node. In contrast to $P_{\text{TSP}}(n)$, it is a classical result in combinatorial optimization that linear functions can be optimized over the matching polytope in polynomial time.

In fact, for both polytopes Yannakakis was able to give superpolynomial lower bounds on sizes of extended formulations that satisfy some symmetry property – however, leaving the question open for general extended formulations. Two decades later, Kaibel, Pashkovich & Theis [48] picked up this problem and gave examples of polytopes for which such symmetry assumptions turned out to be actual restrictions with respect to sizes of extensions. Based on counting arguments, Rothvoß [71] subsequently proved the existence of a family of 0/1-

polytopes (polytopes whose vertices have coordinates in $\{0, 1\}$) whose extension complexities grow exponential in their dimension. Restricting to a more concrete class of polytopes, he further showed that these polytopes can be chosen to be independence polytopes of matroids, which will be the central objects of Section 3.3. Independently, Fiorini et al. [30] developed general, purely combinatorial bounds on sizes of extended formulations. Their ideas were based on connections between extended formulations and certain communication protocols that have been established by Yannakakis [80]. In their breakthrough paper [29], Fiorini et al. used these techniques in order to finally establish a superpolynomial lower bound on $\mathrm{xc}(P_{\mathrm{TSP}}(n))$. We will shed more light on this result in Section 4.2. Two years later, using similar but more involved techniques, Rothvoß [72] was even able to answer Yannakakis' second question by proving that $\mathrm{xc}(P_{\mathrm{match}}(n))$ grows indeed exponentially in n.

Many of the results mentioned here are presented in the survey of Kaibel [44] and the book of Conforti, Cornuéjols & Zambelli [18, Chap. 4]. Further recent developments concern bounds on sizes of *approximate* extended formulations, which are essentially covered by the work in [10, 8, 15, 9, 7]. However, we touch this concept only very briefly in Section 4.1.3.

Outline This chapter is organized as follows. The aim of the first part is to present new polynomial-size extended formulations for several combinatorial polytopes. To this end, we first investigate known methods to derive extended formulations – with a particular emphasis on constructions for the spanning-tree polytope. We introduce the nonempty-subgraph polytope $P^{\star}_{\mathrm{sub}}(G)$ and show that it can be seen as an important brick in Martin's [56] construction of an $\mathcal{O}(|V| \cdot |E|)$ size extended formulation for $P_{\mathrm{sp.trees}}(G)$. More interestingly, we prove that the extension complexities of $P^{\star}_{\mathrm{sub}}(G)$ and $P_{\mathrm{sp.trees}}(G)$ coincide up to low-order terms and on the way give a complete linear description of the former.

In Section 3.3, we present constructions of polynomial-size extended formulations for independence polytopes of several classes of matroids. First, we consider the family of count matroids. As an application of the observations of the previous section, we show how the relationship between $P^{\star}_{\text{sub}}(G)$ and $P_{\text{sp.trees}}(G)$ can be exploited in order to derive polynomial-size extended formulations for independence polytopes of such matroids. This generalizes recent results by Iwata et al. [40]. The second class of matroids we consider is the well-known family of regular matroids. We prove that the extension complexity of the independence polytope of any such matroid grows at most quadratically in the dimension.

In Section 3.4, we show that the extension complexity of the cut dominant can be bounded, up to additive terms of lower order, by the extension complexity of the spanning-tree polytope.

The second part of this chapter deals with limitations of extended formulations. We first revisit general lower bounds on sizes of extended formulations. There, we mainly focus on three well-known bounds on the extension complexity, namely the rectangle-covering bound, the fractional rectangle-covering bound, and the hyperplane-separation bound. For the first two bounds, we give new examples demonstrating their limitations in particular cases: We show that the rectangle covering number of independence polytopes of matroids grows at most quadratic in the dimension. Furthermore, we give an $O(n^{2+2/3} \log n)$-size fractional rectangle covering for $P_{\text{sp.trees}}(n)$. Regarding the hyperplane-separation bound, we show that it can be easily applied to obtain bounds on approximate extended formulations in certain (simple) cases.

As an application of the rectangle-covering bound, in Section 4.2, we give a short, simple and purely combinatorial proof of the fact that the extension complexity of the correlation polytope $P_{\text{corr}}(n)$ is at least 1.5^n. To our knowledge, this turns out to be the best known bound on $\text{xc}(P_{\text{corr}}(n))$. Establishing a superpolynomial lower bound on $\text{xc}(P_{\text{corr}}(n))$ has been the crucial part in the work of Fiorini et al. [29] who showed that the extension complexity of $P_{\text{TSP}}(n)$ grows superpolynomially.

Finally, in Section 4.3, we investigate extended formulations of a special type. Namely, we restrict our attention to extensions in which every vertex of the extension polytope is projected down to a vertex of the image. This property is satisfied by many classical constructions. However, we show that there are polytopes for which in every minimum size extension a certain fraction of its vertices violate this property.

Several parts of this chapter are results of discussions with other colleagues and often form the bases for joint publications: The results of Section 3.2.1 and Section 3.3.2 are presented in

- Michele Conforti, Volker Kaibel, Matthias Walter, and Stefan Weltge. Subgraph polytopes and independence polytopes of count matroids. *Operations Research Letters*, 43(5):457–460, 2015.

The observations made in Section 3.4 originated from discussions with Samuel Fiorini, Volker Kaibel and Kanstantsin Pashkovich. Section 3.3.3 forms the base of

- Volker Kaibel, Jon Lee, Matthias Walter, and Stefan Weltge. Extended Formulations for Independence Polytopes of Regular Matroids. Submitted, arXiv:1504.03872, 2015,

which also contains Proposition 4.1.3. Proposition 4.1.4 was inspired by several discussions with Samuel Fiorini, Kanstantsin Pashkovich and Dirk Oliver Theis. Theorem 4.2.2 is presented in

- Volker Kaibel and Stefan Weltge. A Short Proof that the Extension Complexity of the Correlation Polytope Grows Exponentially. *Discrete & Computational Geometry*, 53(2):396–401, 2015.

The observations of Section 4.2.2 have been made together with Arnaud Vandaele. Section 4.3 forms the base of

- Kanstantsin Pashkovich and Stefan Weltge. Hidden Vertices in Extensions of Polytopes. *Operations Research Letters*, 43(2):161–164, 2015.

3

Describing Polytopes: Constructions

3.1 General Techniques

In this part, we present new polynomial-size extended formulations for several combinatorial polytopes. Many of these constructions are inspired or motivated by Martin's [56] extended formulation for the spanning-tree polytope. In order to analyze his extension, we first need to review general upper bounds on the extension complexity of two simple constructions. However, these observations will also be very useful in our derivation of new extended formulations.

3.1.1 Disjunctions

Given nonempty polytopes $P_1, \ldots, P_k \subseteq \mathbb{R}^d$ we are interested in the convex hull of their union, i.e., $P := \operatorname{conv}(P_1 \cup \cdots \cup P_k)$. In general, having at hand outer descriptions by means of linear inequalities for each P_i, it might be a difficult task to derive an outer description for P. However, in his work on *disjunctive programming* Balas [5] gives a simple extended formulation for P involving the descriptions of the P_i's. Extending his ideas by replacing the outer descriptions of the P_i's by extended formulations, it is possible to bound the extension complexity of P in terms of the extension complexities of $P_1, \dots, P_k$: If each P_i can be described via

$$P_i = \left\{ p^i(z) + \gamma^i \mid z \in \mathbb{R}^{d_i},\, A^i z \le b^i \right\}$$

for some $A^i \in \mathbb{R}^{m_i \times d_i}$, $b^i \in \mathbb{R}^{m_i}$, $\gamma^i \in \mathbb{R}^d$ and a linear map $p^i \colon \mathbb{R}^{d_i} \to \mathbb{R}^d$, then an extended formulation for P is given by

$$\begin{aligned} P = \Bigg\{ \sum_{i=1}^{k} p^i(y^i) + \sum_{i=1}^{k} \gamma^i \lambda_i \mid A^i y^i \le \lambda_i \cdot b^i \; i = 1, \dots, k, \\ \sum_{i=1}^{k} \lambda_i = 1 \text{ and } \lambda \ge \mathbb{O} \Bigg\}. \end{aligned} \tag{3.1}$$

A proof of this fact can be found in [65]. In particular, the extension complexity of $\operatorname{conv}(\bigcup_{i=1}^{k} P_i)$ is at most $\sum_{i=1}^{k}(\operatorname{xc}(P_i) + 1)$. We show that this general bound can be slightly improved.

Proposition 3.1.1. *Let $P_1, \dots, P_k \subseteq \mathbb{R}^d$ be polytopes. Then we have*

$$\operatorname{xc}\Big(\operatorname{conv}(P_1 \cup \dots \cup P_k) \Big) \le \sum_{i=1}^{k} \operatorname{xc}(P_i) + \Big| \{ i \in [k] \mid \dim(P_i) = 0 \} \Big|.$$

Proof. We use the notation from the preceding paragraph, where we require that each system $A^i z \le b^i$ provides an extended formulation for P_i that has minimum size. We may further assume that the polyhedra $\{z \in \mathbb{R}^{d_i} \mid A^i z \le b^i\}$ are bounded, see, e.g., the discussion in the introduction of [30]. Let us now fix some $i \in \{1, \dots, k\}$ such that $\dim(P_i) > 0$ holds. Given the extended formulation in (3.1), it suffices to show that $A^i y^i \le \lambda_i \cdot b^i$ for some $y^i \in \mathbb{R}^{d_i}$, $\lambda_i \in \mathbb{R}$ already implies $\lambda_i \ge 0$. Equivalently, we have to show that the face $F := \{(z, 0) \in \mathbb{R}^{d_i+1} \mid A^i z \le 0 \cdot b^i\}$ of the polyhedron $Q := \{(z, \lambda_i) \mid A^i z \le \lambda_i \cdot b^i,\, \lambda_i \ge 0\}$ is not a facet of Q. To this end, first observe that we have

$$\dim(F) = \dim(\{z \in \mathbb{R}^{d_i} \mid A^i z \le \mathbb{O}\}) = \dim(\{\mathbb{O}\}) = 0,$$

where the second equality holds since $\{z \in \mathbb{R}^{d_i} \mid A^i z \le b^i\}$ is bounded. On the other hand, we have

$$\dim(Q) = \dim(\{z \in \mathbb{R}^{d_i} \mid A^i z \le b^i\}) + 1 \ge \dim(P_i) + 1 \ge 2,$$

and hence F indeed cannot be a facet of Q. $\square$

3.1.2 LP-Duality

The second construction we consider is due to Martin [56] and requires two polyhedra to share a certain property that can be seen as a generalization of usual polarity.

Proposition 3.1.2. *Given a nonempty polyhedron Q and $\gamma \in \mathbb{R}$, let*

$$P = \{x \mid \langle x, y\rangle \le \gamma \textit{ for all } y \in Q\}\,.$$

If $Q = \{y \mid \exists z : Ay + Bz \le b, Cy + Dz = d\}$, then we have that

$$\begin{aligned} P = \{x \mid \exists \lambda \ge \mathbb{O}, \exists \mu : A^\intercal \lambda + C^\intercal \mu = x,\\ B^\intercal \lambda + D^\intercal \mu = \mathbb{O}, \langle b, \lambda\rangle + \langle d, \mu\rangle \le \gamma] \end{aligned}$$

holds and hence $\mathrm{xc}(P) \le \mathrm{xc}(Q) + 1$.

Proof. A point $\bar{x}$ is contained in P if and only if

$$\max\,\{\langle \bar{x}, y\rangle \mid \exists z : Ay + Bz \le b, Cy + Dz = d\} \le \gamma,$$

which by strong duality is equivalent to the existence of dual multipliers $\lambda \ge \mathbb{O}$ and (unconstrained) μ such that $A^\intercal \lambda + C^\intercal \mu = \bar{x}$, $B^\intercal \lambda + D^\intercal \mu = \mathbb{O}$, and $\langle b, \lambda\rangle + \langle d, \mu\rangle \le \gamma$ hold. □

The above statement can be seen as one of the key observations in Martin's [56] construction of an extended formulation for the spanning-tree polytope, which we will revisit in the next section in a slightly more abstract manner than given in his paper.

3.2 Spanning-Tree Polytope

Given a connected undirected graph $G = (V, E)$, the spanning-tree polytope $P_{\text{sp.trees}}(G)$ is defined as

$$P_{\text{sp.trees}}(G) := \operatorname{conv}\left\{\chi(F) \in \{0,1\}^E \mid F \subseteq E \text{ spanning tree}\right\}.$$

It is a classical fact due to Edmonds [23] that $P_{\text{sp.trees}}(G)$ equals the set of points $x \in \mathbb{R}^E_{\ge 0}$ satisfying $x(E) = |V| - 1$ and $x(F) \le |S| - 1$ for all $F \subseteq E(S)$ with $\emptyset \ne S \subseteq V$. Here, $E(S)$ denotes the set of edges in E that have both endnodes in S. Alternatively, a point $x \in \mathbb{R}^E_{\geq 0}$ with $x(E) = |V| - 1$ is contained in $P_{\text{sp.trees}}(G)$ if and only if

$$\Big\langle (x, -\mathbb{1}_V), (\chi(F), \chi(S)) \Big\rangle \leq -1 \quad \text{for all } F \subseteq E(S),\ \emptyset \neq S \subseteq V$$

holds. Thus, defining the polytope

$$P^\star_{\text{sub}}(G) := \operatorname{conv}\Big\{ (\chi(F), \chi(S)) \in \{0,1\}^E \times \{0,1\}^V \mid F \subseteq E(S),\ \emptyset \neq S \subseteq V \Big\}$$

we obtain

$$P_{\text{sp.trees}}(G) = \Big\{ x \in \mathbb{R}^E_{\geq 0} \mid x(E) = |V| - 1, \\ \Big\langle (x, -\mathbb{1}_V), (y, z) \Big\rangle \leq -1 \text{ for all } (y, z) \in P^\star_{\text{sub}}(G) \Big\}$$

and hence, by Proposition 3.1.2,

$$\operatorname{xc}(P_{\text{sp.trees}}(G)) \leq \operatorname{xc}(P^\star_{\text{sub}}(G)) + |E| + 1. \tag{3.2}$$

The next section is dedicated to the investigation of $P^\star_{\text{sub}}(G)$, which we call the *nonempty-subgraph polytope* of G.

3.2.1 Nonempty-Subgraph Polytope

First, we establish a simple upper bound on the extension complexity of $P^\star_{\text{sub}}(G)$. To this end, let us consider the *subgraph polytope* of a connected graph G defined via

$$P_{\text{sub}}(G) := \operatorname{conv}\Big\{ (\chi(F), \chi(S)) \in \{0,1\}^E \times \{0,1\}^V \mid F \subseteq E(S),\ S \subseteq V \Big\}.$$

A system of valid linear inequalities whose set of feasible integer points coincides with the set of integer points in $P_{\text{sub}}(G)$ is given by

$$0 \leq z_v \leq 1 \quad \text{for all } v \in V, \tag{3.3}$$

$$0 \leq y_{\{v,w\}} \leq z_v \quad \text{for all } \{v, w\} \in E. \tag{3.4}$$

Let A be the coefficient matrix describing system (3.3), (3.4). Since A has at most one $+1$ and one -1 in each row, A is totally unimodular and hence

$$P_{\text{sub}}(G) = \Big\{ (y, z) \in \mathbb{R}^E \times \mathbb{R}^V \mid (y, z) \text{ satisfies (3.3), (3.4)} \Big\}.$$

Now observe that $P^\star_{\text{sub}}(G)$ arises from $P_{\text{sub}}(G)$ by removing the single vertex $(\mathbb{O}_E, \mathbb{O}_V)$. For each $v \in V$, let Q_v be the face of $P_{\text{sub}}(G)$ that is defined by $z_v = 1$. With this notation, we clearly have $P^\star_{\text{sub}}(G) = \text{conv}(\cup_{v\in V} Q_v)$ and hence finally obtain

$$\text{xc}(P^\star_{\text{sub}}(G)) \leq \sum_{v\in V}(\text{xc}(Q_v) + 1) \leq \sum_{v\in V}(\text{xc}(P_{\text{sub}}(G)) + 1) \leq \mathcal{O}(|V| \cdot |E|). \quad (3.5)$$

Together with Inequality (3.2) we summarize

$$\text{xc}(P_{\text{sp.trees}}(G)) \leq \text{xc}(P^\star_{\text{sub}}(G)) + |E| + 1 \leq \mathcal{O}(|V| \cdot |E|). \quad (3.6)$$

It is an open question whether this bound on the extension complexity of the spanning tree polytope is tight for general graphs. For the case of planar graphs a result by Williams [79] provides an extended formulation for $P_{\text{sp.trees}}(G)$ whose size is even linear. One might ask whether the bound given in (3.5) is best possible for general graphs. Note that the extended formulation behind this inequality is a special case of those constructed in [2], where the general problem of removing vertices from polytopes is investigated.

Clearly, any construction yielding an asymptotically smaller extension for $P^\star_{\text{sub}}(G)$ would imply an improved upper bound on the extension complexity of the spanning-tree polytope. In what follows, we will show that also the converse holds. To this end, we first give a complete description of $P^\star_{\text{sub}}(G)$ in the original space.

Theorem 3.2.1. *For a connected undirected graph $G = (V, E)$ we have*

$$P^\star_{\text{sub}}(G) = P_{\text{sub}}(G) \cap \{(y, z) \in \mathbb{R}^E \times \mathbb{R}^V \mid y(T) \leq z(V) - 1 \textit{ for all spanning trees } T \subseteq E\}.$$

Proof. Let Q denote the polytope on the right-hand side of the equation. It is easy to check that the inequalities defining Q imply $z(V) \geq 1$ and are valid for all vertices of $P_{\text{sub}}(G)$ except the origin. Thus, the integer points in $P^\star_{\text{sub}}(G)$ and Q coincide and it suffices to show that Q has only integer vertices.

First, suppose that we have a point (y, z) that satisfies (3.3) and (3.4) with $z_v = 1$ for some $v \in V$. Given a spanning tree $T \subseteq E$, inequalities (3.4) together with the nonnegativity of z imply

$$y(T) \leq z(V \setminus \{v\}) = z(V) - z_v = z(V) - 1.$$

Thus, every face of Q defined by $z_v = 1$ for some $v \in V$ coincides with the face of $P_{\text{sub}}(G)$ defined by $z_v = 1$ and hence has only integer vertices.

Let (y, z) be any vertex of Q. It remains to show that this implies $z_v = 1$ for some $v \in V$. For the sake of contradiction, assume that we have $z_v < 1$ for all $v \in V$. By possibly deleting nodes and edges of G, we may assume that we have $z_v \geq y_{\{v,w\}} > 0$ for all $\{v, w\} \in E$. Then (y, z) is the unique solution of a system

$$y_{\{v,w\}} = z_v \qquad \text{for all } \{v, w\} \in E' \tag{3.7}$$

$$y(F) = z(V) - 1 \qquad \text{for all } F \in \mathcal{F} \tag{3.8}$$

of linear equations for some $E' \subseteq E$ and some nonempty collection $\mathcal{F}$ of spanning forests in G. Let $\alpha := \max_{e\in E} y_e$ and set $\overline{E} := \{e \in E \mid y_e = \alpha\}$. Let (V_α, E_α) be a connected component of $(V, \overline{E})$ containing at least one edge and let us define $(y', z') \in \mathbb{R}^E \times \mathbb{R}^V$ as follows:

$$y'_e := \begin{cases} 2 \cdot y_e & \text{if } e \in E \setminus E(V_\alpha), \\ 2 \cdot y_e - 1 & \text{if } e \in E(V_\alpha), \end{cases}$$

$$z'_v := \begin{cases} 2 \cdot z_v & \text{if } v \in V \setminus V_\alpha, \\ 2 \cdot z_v - 1 & \text{if } v \in V_\alpha. \end{cases}$$

As we have $y_{\{v,w\}} < \alpha \leq z_v$ if $\{v, w\} \notin E_\alpha$ and $v \in V_\alpha$, we obtain that (y', z') satisfies (3.7). Let F^* be a spanning forest such that $y(F^*) = z(V) - 1$ holds. Since $y(F) \leq z(V) - 1$ for every spanning forest F, we have that F^* is a spanning forest of maximum y-weight. Following Kruskal's algorithm we find that $|F^* \cap E_\alpha| = |V_\alpha| - 1$ holds, and hence

$$\begin{aligned} y'(F^*) &= 2y(F^*) - (|V_\alpha| - 1) \\ &= 2(z(V) - 1) - (|V_\alpha| - 1) \\ &= 2z(V) - |V_\alpha| - 1 \\ &= z'(V) - 1. \end{aligned}$$

Since $0 < z_v < 1$ holds for all $v \in V$, we have $(y', z') \neq (y, z)$. Therefore (y', z') is another solution to the system (3.7)–(3.8) and this contradicts the fact that (3.7)–(3.8) defines a vertex of Q. □

Using Proposition 3.1.2, the above statement implies that every extended formulation for $P_{\text{sp.trees}}(G)$ can be transferred into one for $P^{\star}_{\text{sub}}(G)$ of essentially the same size.

Theorem 3.2.2. *The extension complexities of $P_{\text{sp.trees}}(G)$ and $P^{\star}_{\text{sub}}(G)$ coincide up to an additive term of order $O(|E|)$.*

Proof. By Inequality (3.2), we already have that

$$\operatorname{xc}(P_{\text{sp.trees}}(G)) \le \operatorname{xc}(P^{\star}_{\text{sub}}(G)) + O(|E|)$$

holds. Setting

$$P := \Big\{(y,z) \in \mathbb{R}^E \times \mathbb{R}^V \mid y(T) - z(V) \le -1 \text{ for all spanning trees } T \subseteq E\Big\},$$
$$Q := P_{\text{sp.trees}}(G) \times \{-\mathbb{1}_V\},$$

and $\gamma := -1$, we obtain

$$\begin{aligned}
\operatorname{xc}(P^{\star}_{\text{sub}}(G)) &= \operatorname{xc}(P_{\text{sub}}(G) \cap P)\\
&\le \operatorname{xc}(P_{\text{sub}}(G)) + \operatorname{xc}(P)\\
&\le \operatorname{xc}(P_{\text{sub}}(G)) + \operatorname{xc}(Q) + 1\\
&= \operatorname{xc}(P_{\text{sub}}(G)) + \operatorname{xc}(P_{\text{sp.trees}}(G)) + 1\\
&\le O(|E|) + \operatorname{xc}(P_{\text{sp.trees}}(G)),
\end{aligned}$$

where the first equality follows from Theorem 3.2.1, the second inequality from Proposition 3.1.2, and the third inequality from the outer description of $P_{\text{sub}}(G)$ given by (3.3) and (3.4). □

3.3 Independence Polytopes of Matroids

In this section, we consider independence polytopes of matroids and show how the observations from the previous sections can be applied to obtain new polynomial-size extended formulations for several classes of these polytopes. Recall that a *matroid* is a pair $(E, \mathcal{I})$, where E is a finite set (called the *ground set*) and $\mathcal{I}$ is a nonempty collection of subsets of E such that (i) for every $F' \subseteq F$ with $F \in \mathcal{I}$ we also have that $F' \in \mathcal{I}$ holds, and (ii) for every two sets $A, B \in \mathcal{I}$ with $|A| < |B|$

there exists an element $b \in B \setminus A$ with $A \cup \{b\} \in \mathcal{I}$. The sets in $\mathcal{I}$ are called *independent sets*. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the *independence polytope* of $\mathcal{M}$ is defined as

$$P(\mathcal{M}) := \operatorname{conv}\{\chi(I) \mid I \in \mathcal{I}\}.$$

Independence polytopes of matroids are central objects in the field of combinatorial optimization. It is well-known that all non-trivial facet-defining inequalities for $P(\mathcal{M})$ are of the form $\sum_{i \in S} x_i \leq r(S)$ with $S \subseteq E$, where r denotes the rank function of $\mathcal{M}$. Furthermore, any linear function can be maximized over $P(\mathcal{M})$ by a simple greedy algorithm involving only a linear number of independence oracle calls. For more results on independence polytopes of matroids, see, e.g., Schrijver [74].

In this sense, independence polytopes of matroids are well understood and seem to be simple objects. One might wonder whether all such polytopes admit polynomial-size extended formulations. Unfortunately, this question was answered negatively by Rothvoß [71] who showed that there exists a family of independence polytopes of matroids having extension complexity growing exponentially in the dimension.

On the positive side, there are only a few interesting classes of matroids for which we know that the corresponding independence polytopes admit polynomial size extensions. As we will see, using the extended formulation for the spanning-forest polytope presented in the previous section, it is easy to derive quadratic size extended formulations for independence polytopes of graphic and cographic matroids. Recently, this has been generalized by Iwata et al. [40] to the class of sparsity matroids. In Section 3.3.2, we give an alternative, simple proof of this fact and extend their results to the even more general class of count matroids. The second class of matroids we consider is the family of regular matroids – another, well-known superclass of (co-)graphic matroids. In Section 3.3.3, we prove that the extension complexity of the independence polytope of any such matroid grows at most quadratically in the dimension.

3.3.1 Graphic & Cographic Matroids

Classic examples of matroids are *graphic matroids*. Given an undirected graph $G = (V, E)$, the graphic matroid of G has ground set E, where a set of edges is independent if and only if it does not contain a cycle. In other words, if G is connected, then a set of edges is independent if it is contained in a spanning tree. Thus, if $\mathcal{M}(G)$ is the graphic matroid of some connected graph G, we have

$$P(\mathcal{M}(G)) = \operatorname{conv}\left\{x \in \{0,1\}^E \mid x \leq \chi(T) \text{ for some spanning tree } T\right\}. \tag{3.9}$$

In order to derive an extended formulation for $P(\mathcal{M}(G))$, we make use of the following simple result concerning the monotonization of 0/1-polytopes. Though it can probably be considered folklore, we include a brief proof, since we are not aware of any appropriate reference.

Lemma 3.3.1. *For* $Y \subseteq \{0,1\}^n$, $P := \operatorname{conv}(Y)$, *and*

$$P^{\downarrow} := \operatorname{conv}\{x \in \{0,1\}^n \mid x \leq y \textit{ for some } y \in Y\},$$

we have

$$P^{\downarrow} = \left\{x \in \mathbb{R}^n_{\geq 0} \mid \exists y \in P : x \leq y\right\},$$

thus $\operatorname{xc}(P^{\downarrow}) \leq \operatorname{xc}(P) + 2n$.

Proof. Let us define $Q := \left\{x \in \mathbb{R}^n_{\geq 0} \mid \exists x \in P : y \leq x\right\}$. For every $c \in \mathbb{R}^n$, setting $\bar{c}_i := \max\{c_i, 0\}$ for all $i \in [n]$ yields

$$\begin{aligned}
\max\left\{\langle c, x\rangle \mid x \in P^{\downarrow}\right\} &= \max\left\{\langle c, x\rangle \mid x \in \{0,1\}^n,\, x \leq y \text{ for some } y \in Y\right\} \\
&= \max\left\{\langle \bar{c}, y\rangle \mid y \in Y\right\} \\
&= \max\left\{\langle \bar{c}, y\rangle \mid y \in P\right\} \\
&= \max\left\{\langle c, x\rangle \mid x \in Q\right\},
\end{aligned}$$

and thus $P^{\downarrow} = Q$. □

For connected graphs G, Lemma 3.3.1 and Equation (3.9) imply

$$\operatorname{xc}(\mathcal{M}(G)) \leq \operatorname{xc}(P_{\text{sp.trees}}(G)) + 2|E|. \tag{3.10}$$

Closely related to graphic matroids are *cographic matroids*, which are the duals of graphic matroids. The cographic matroid of an undirected graph $G = (V, E)$ also has ground set E, where now a set of edges is independent if and only if it is contained in the complement of a spanning forest. Thus, if $\mathcal{M}^*(G)$ is the cographic matroid of some connected graph G, we obtain

$$P(\mathcal{M}^*(G)) = \operatorname{conv}\left\{x \in \{0,1\}^E \mid x \leq \mathbb{1} - \chi(T) \text{ for some spanning tree } T\right\}.$$

Hence, again by Lemma 3.3.1, this implies

$$\begin{aligned} \operatorname{xc}(P(\mathcal{M}^*(G))) &\leq \operatorname{xc}(\mathbb{1} - P_{\text{sp.trees}}(G)) + 2|E| \\ &= \operatorname{xc}(P_{\text{sp.trees}}(G)) + 2|E|. \end{aligned} \tag{3.11}$$

Given a connected undirected graph $G = (V, E)$, by Inequality (3.6) we have that the extension complexity of $P_{\text{sp.trees}}(G)$ can be bounded by $\mathcal{O}(|V| \cdot |E|)$, which can be further estimated by $\mathcal{O}(|E|^2)$ because one has $|E| \geq |V| - 1$. Finally, observe that the independence polytope of the (co-)graphic matroid of a graph is the Cartesian product of the independence polytopes of the (co-)graphic matroids of its connected components. Thus, using Inequality (3.10) and Inequality (3.11), we conclude:

Proposition 3.3.2. *For any graphic or cographic matroid $\mathcal{M}$ on ground set E we have*

$$\operatorname{xc}(P(\mathcal{M})) \leq \mathcal{O}(|E|^2).$$

3.3.2 Count Matroids

Let $G = (V, E)$ be an undirected graph, $\ell \in \mathbb{Z}_{\geq 0}$ be some non-negative integer and $m \colon V \to \mathbb{Z}_{\geq 0}$ be a non-negative integer valued function satisfying

$$m(v) + m(w) \geq \ell \quad \forall \{v, w\} \in E. \tag{3.12}$$

Consider the independence system $\mathcal{M}_{m,\ell}(G)$ on ground set E where a set $F \subseteq E$ is independent if and only if

$$|F \cap E(S)| \leq \max\{m(S) - \ell, 0\}$$

holds for all $S \subseteq V$, where $m(S) := \sum_{v \in S} m(v)$. Such independence systems can be easily seen to satisfy the matroid axioms and are called *count matroids*, see [31]. If we have $m(v) = k$ for all $v \in V$ for some $k \in \mathbb{Z}$, the matroid $\mathcal{M}_{k,\ell}(G) := \mathcal{M}_{m,\ell}(G)$ is called a *(k, ℓ)-sparsity matroid*. Note that the $(1,1)$-sparsity matroid of G is simply the graphic matroid of G. A theorem of Nash-Williams [60] states that the independent sets of the (k,k)-sparsity matroid of G are those subsets of edges of E that can be partitioned into k forests.

Recently, Iwata et al. [40] showed the existence of polynomial size extended formulations for independence polytopes of (k,ℓ)-sparsity matroids. More precisely, they showed that the extension complexity of $P(\mathcal{M}_{k,\ell})$ can be bounded by $\mathcal{O}(|V|\cdot|E|)$ if $k \geq \ell$, and by $\mathcal{O}(|V|^2\cdot|E|)$ otherwise. They employed a technique developed in [27] and designed a randomized communication protocol (exchanging only few bits) that computes the slack matrix of these polytope in expectation. This approach defines an extended formulation only implicitly. It probably would be a rather tedious task to explicitly derive an extended formulation from that protocol, which consequently is not done in [40].

In what follows, we give polynomial bounds on the extension complexities of independence polytopes of count matroids, which match the ones given in [40] for the special case of sparsity matroids. Our proof technique is based on the observations in Section 3.2 and allows to easily work out explicit extended formulations. In addition, we are even able to improve upon the bounds given in [40] in some cases if the underlying graph is planar.

In our argumentation, we have to distinguish two cases. In the first case, we assume that m,ℓ satisfy the following additional requirement:

$$m(v) \geq \ell \quad \forall\, v \in V. \tag{3.13}$$

Note that this case corresponds to the assumption $k \geq \ell$ in the case of (k,ℓ)-sparsity matroids. The second case deals with the general situation in which (3.13) is not necessarily satisfied.

Theorem 3.3.3. *Let $G=(V,E)$ be a connected undirected graph and $\mathcal{M}_{m,\ell}(G)$ be a count matroid satisfying (3.13). Then we have*

(a) $\operatorname{xc}(P(\mathcal{M}_{m,\ell}(G)) \leq \mathcal{O}(|V|\cdot|E|)$,

(b) $\operatorname{xc}(P(\mathcal{M}_{m,\ell}(G)) \leq \mathcal{O}(|E|)$ *if G is planar.*

Proof. Due to condition (3.13), $P(\mathcal{M}_{m,\ell}(G))$ can be described via

$$P(\mathcal{M}_{m,\ell}(G)) = \left\{x \in \mathbb{R}^E_{\geq 0} \mid x(F) \leq m(S) - \ell \text{ for all } F \subseteq E(S),\ \emptyset \neq S \subseteq V\right\}$$

or, alternatively,

$$P(\mathcal{M}_{m,\ell}(G)) = \left\{x \in \mathbb{R}^E_{\geq 0} \mid \left\langle (x, -m)), (y, z) \right\rangle \leq -\ell \text{ for all } (y, z) \in P^\star_{\text{sub}}(G)\right\}.$$

Thus, via Proposition 3.1.2 and Inequality (3.5) we conclude

$$\operatorname{xc}(P(\mathcal{M}_{m,\ell}(G))) \leq \operatorname{xc}(P^\star_{\text{sub}}(G)) + |E| + 1 \leq \mathcal{O}(|V| \cdot |E|),$$

which shows (a). By Theorem 3.2.2 this further implies

$$\operatorname{xc}(P(\mathcal{M}_{m,\ell}(G))) \leq \operatorname{xc}(P_{\text{sp.trees}}(G)) + \mathcal{O}(|E|),$$

and hence (b) follows from Williams' linear size extended formulation for $P_{\text{sp.trees}}(G)$ in the case of planar graphs [79]. □

In the above proof, we have actually seen that

$$\operatorname{xc}(P(\mathcal{M}_{m,\ell}(G)) \leq \operatorname{xc}(P_{\text{sp.trees}}(G)) + \mathcal{O}(|E|)$$

holds. Recall that $P_{\text{sp.trees}}(G)$ is a face of $P(\mathcal{M}_{m,\ell}(G))$ with $m = \mathbb{1}_V$ and $\ell = 1$. Thus, among all (m, ℓ)-count matroids with m, ℓ satisfying (3.13), up to additive terms of order $\mathcal{O}(|E|)$, an upper bound on the extension complexities of their independence polytopes is already attained for the choice $(m, \ell) = (\mathbb{1}_V, 1)$.

The proof of Theorem 3.3.3 uses the fact that if m, ℓ satisfy (3.13), then the inequality $x(F) \leq m(S) - \ell$ with $F \subseteq E(S)$ is even valid for $P(\mathcal{M}_{m,\ell}(G))$ if S consists of a single node. This allowed us to use $P^\star_{\text{sub}}(G)$ to describe $P(\mathcal{M}_{m,\ell}(G))$. For the general case, we have to make use of another polytope related to $P_{\text{sub}}(G)$.

Theorem 3.3.4. *Let $G = (V, E)$ be a connected undirected graph and $\mathcal{M}_{m,\ell}(G)$ be any count matroid. Then we have*

$$\operatorname{xc}(P(\mathcal{M}_{m,\ell}(G))) \leq \mathcal{O}(|E|^2).$$

Proof. Let us consider the polytope

$$P^{\star\star}_{\text{sub}}(G) := \operatorname{conv}\left\{(\chi(F), \chi(S)) \mid F \subseteq E(S),\ e \subseteq S \subseteq V,\ e \in E\right\}.$$

Defining the face Q_e of $P_{\text{sub}}(G)$ for each edge $e = \{v, w\} \in E$ via

$$Q_e := \left\{(y, z) \in P_{\text{sub}}(G) \mid z_v = z_w = 1\right\},$$

we clearly have $P^{\star\star}_{\text{sub}}(G) = \operatorname{conv}(\cup_{e\in E} Q_e)$ and thus

$$\operatorname{xc}(P^{\star\star}_{\text{sub}}(G)) \leq \sum_{e\in E} (\operatorname{xc}(Q_e) + 1) \leq O(|E| \cdot |E|).$$

Due to (3.12), $P(\mathcal{M}_{m,\ell}(G))$ can be described via

$$P(\mathcal{M}_{m,\ell}(G)) = \Big\{ x \in \mathbb{R}^E_{\geq 0} \mid x(F) \leq m(S) - \ell \text{ for all } F \subseteq E(S) \\ \text{with } e \subseteq S \subseteq V \text{ for some } e \in E \Big\}$$

or, alternatively,

$$P(\mathcal{M}_{m,\ell}(G)) = \Big\{ x \in \mathbb{R}^E_{\geq 0} \mid \Big\langle (x, -m), (y, z) \Big\rangle \leq -\ell \text{ for all } (y, z) \in P^{\star\star}_{\text{sub}}(G) \Big\},$$

and hence the claim follows from Proposition 3.1.2. □

In contrast to the polytope $P^{\star}_{\text{sub}}(G)$, from computer experiments it seems that the polytope $P^{\star\star}_{\text{sub}}(G)$ used in the proof of Theorem 3.3.4 has a very complicated facet structure. In fact, we do not even have a conjecture how an inequality description in the original space could look like.

3.3.3 Regular Matroids

Another, well-known family of matroids comprising graphic and cographic matroids is the class of *linear matroids*. Given a matrix $A \in \mathbb{F}^{p\times q}$ with entries in some field $\mathbb{F}$, we denote by

$$\mathcal{M}_{\mathbb{F}}(A) := \{ I \subseteq \{1, \dots, p+q\} \mid (\mathbb{I}, A)_{*,I} \text{ has full column-rank over } \mathbb{F} \}$$

the (set of independent sets of the) matroid defined by A, where $\mathbb{I}$ is the $p \times p$-identity matrix. Therefore, the cardinality of the ground set of $\mathcal{M}_{\mathbb{F}}(A)$ is $p + q$, i.e., the number columns of the *identity-extension* $(\mathbb{I}, A)$ of the matrix A. Note that this use of notation differs, e.g., from Oxley [62], but is in accordance to Schrijver's book [73, Chap. 19]. Here, we allow A to have $q = 0$ columns (in which case $\mathcal{M}_{\mathbb{F}}(A)$ is a *free matroid* on p elements with all subsets being independent) or $p = 0$ rows (in which case $\mathcal{M}_{\mathbb{F}}(A)$ is a matroid on q elements such that the empty set is the only independent set), but we will always have $p + q > 0$. If a matroid $\mathcal{M}$ is isomorphic to $\mathcal{M}_{\mathbb{F}}(A)$ for some matrix A and some field $\mathbb{F}$, we say that $\mathcal{M}$ can be *represented* (by A) over $\mathbb{F}$. The class of linear matroids consists of all matroids that can be represented over some field.

In this part, we focus on the well-known class of matroids that can be represented over every field, namely *regular matroids*. It can be shown that a matroid is regular if and only if it can be represented by a totally-unimodular matrix over $\mathbb{R}$ (see, e.g., [73, Chap. 19]). A real matrix is said to be *totally unimodular* if every square non-singular sub-matrix has determinant -1 or 1. In particular, any totally-unimodular matrix has only -1, 0 or 1-entries. Note that for a totally-unimodular matrix A, the matroid $\mathcal{M}_{\mathbb{F}}(A)$ does not depend on the specific choice of the field $\mathbb{F}$. Thus, in what follows, we will mainly work over the most simple field $\mathbb{F}_2$, with two elements.

Key examples of regular matroids are graphic matroids. Let $G = (V, E)$ be a connected undirected graph. Choosing some $T \subseteq E$ that forms a spanning tree of G and assigning some orientation to all edges in E, let us construct a matrix $A \in \{0, 1, -1\}^{T \times E}$ as follows: For every pair of (directed) edges $t \in T$ and $e = (v, w) \in E$, set the entry $A_{t,e}$ to 1 or -1 if the path from v to w in T passes through t in forward or backward direction, respectively, and to 0 if it does not pass through t at all. It can be shown that $\mathcal{M}(G)$ is (isomorphic to) $\mathcal{M}_{\mathbb{R}}(A)$ and that A is totally unimodular. In particular, this implies that $\mathcal{M}(G)$ is regular. In a similar way, one can show that cographic matroids are also regular, see, e.g. [78, Sect. 3.2].

Not every regular matroid is graphic or cographic. However, it turns out that all remaining regular matroids can be constructed from only graphic matroids, cographic matroids and matroids of size at most ten. In the following, we formulate this statement more precisely and show how it can be used in order to derive polynomial-size extended formulations for independence polytopes of regular matroids.

Seymour's Decomposition Theorem

From its basic definition, there seems to be no hint concerning crucial properties of regular matroids to exploit in order to obtain polynomial-size extended formulations for the corresponding independence polytopes. Fortunately, there is a very strong structural characterization of regular matroids due to Seymour that we will make use of. In order to state his result, we need to define a few operations on regular matroids. In what follows, all matrices and operations are considered over $\mathbb{F}_2$. For convenience, we will therefore write $\mathcal{M}(A) := \mathcal{M}_{\mathbb{F}_2}(A)$.

Let $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ be binary matroids, i.e., matroids represented over $\mathbb{F}_2$. We say that $\mathcal{M}$ is a *1-sum* of $\mathcal{M}_1$ and $\mathcal{M}_2$ if there exist matrices A, B such that

$$\mathcal{M}_1 = \mathcal{M}(A), \qquad \mathcal{M}_2 = \mathcal{M}(B), \qquad \mathcal{M} = \mathcal{M}\begin{pmatrix} A & \mathbb{O} \\ \mathbb{O} & B \end{pmatrix}$$

holds. We say that $\mathcal{M}$ is a *2-sum* of $\mathcal{M}_1$ and $\mathcal{M}_2$ if there exist matrices A, B and column vectors a, b such that

$$\mathcal{M}_1 = \mathcal{M}\begin{pmatrix} a & A \end{pmatrix}, \qquad \mathcal{M}_2 = \mathcal{M}\begin{pmatrix} b^\intercal \\ B \end{pmatrix}, \qquad \mathcal{M} = \mathcal{M}\begin{pmatrix} A & ab^\intercal \\ \mathbb{O} & B \end{pmatrix}$$

holds. Finally, we say that $\mathcal{M}$ is a *3-sum* of $\mathcal{M}_1$ and $\mathcal{M}_2$ if there exist matrices A, B and column vectors a, b, c, d such that

$$\mathcal{M}_1 = \mathcal{M}\begin{pmatrix} a & a & A \\ 0 & 1 & c^\intercal \end{pmatrix}, \quad \mathcal{M}_2 = \mathcal{M}\begin{pmatrix} 0 & 1 & b^\intercal \\ d & d & B \end{pmatrix}, \quad \mathcal{M} = \mathcal{M}\begin{pmatrix} A & ab^\intercal \\ dc^\intercal & B \end{pmatrix}$$

holds.

In each of the above definitions, we allow A to consist of no columns and B to consist of no rows. Seymour's characterization of regular matroids yields the following:

Theorem 3.3.5 (Regular-Matroid Decomposition Theorem [75], see [73, Thm. 19.6])**.** *For every regular matroid $\mathcal{M}$ there exists a rooted binary tree T whose nodes are binary matroids such that*

- *the root of T is $\mathcal{M}$,*
- *each non-leaf node of T is a k-sum of its two children for some $k \in \{1, 2, 3\}$,*
- *each leaf of T is a graphic matroid, a cographic matroid or has size at most ten,*

- *each leaf of T has a ground set of cardinality at least three, and*
- *whenever a non-leaf node of T is a 3-sum of its children, both children have ground sets of cardinality at least seven.*

In Schrijver's book [73], the above statement is formulated in terms of a decomposition theorem for totally-unimodular matrices, for which he allows certain additional operations on matrices such as adding an all-zero row or column to a matrix A, adding a unit-vector as a row or column to A, or repeating a row or column of A. However, it can be easily seen that all these operations can be interpreted as 2-sums of $\mathcal{M}(A)$ with certain matroids on ground sets of cardinality three. Furthermore, in our definitions of k-sums, we only require the *existence* of matrices defining the components of the sum. Hence, we do not have to allow row/column permutations, pivots or scaling (by -1) of rows or columns. Finally, we also do not need the transposition of a matrix as a particular operation: Given some decomposition tree T as in Theorem 3.3.5, suppose we want to transpose a matrix defining a matroid $\mathcal{M}$ at some non-leaf node of T. By our definition of k-sums, this can be simulated by simply transposing the matrices defining $\mathcal{M}_1$ and $\mathcal{M}_2$ (and swapping the summands). Thus, the transposition can be propagated down to the leaf nodes of T resulting in a decomposition tree $\tilde{T}$, where we replaced some matrices defining leaves by their transposes. Note that the sums of the cardinalities of the ground sets of leaf nodes of T and $\tilde{T}$ coincide. Furthermore, since the transpose of a matrix corresponds to taking the dual of the induced matroid, $\tilde{T}$ is still a decomposition tree as in Theorem 3.3.5.

Our construction

As described in Section 3.3.1, we already have small extended formulations for the independence polytopes of the leaf nodes in decompositions as in Theorem 3.3.5. (Note that the leaf nodes that are not graphic or cographic have size bounded by some constant.) Given a decomposition tree, we are going to to construct an extended formulation for the independence polytope of the root node whose size is small in terms of the sum of the sizes of the extended formulations of the independence polytopes of the leaf nodes. In order to deduce our main result from such a construction later, we first bound the sizes of components in decomposition trees.

Lemma 3.3.6. *Let $\mathcal{M}$ be a regular matroid on ground set E, and let T be a decomposition tree of $\mathcal{M}$ as in Theorem 3.3.5. Then the sum of the cardinalities of the ground sets of leaf nodes of T can be bounded linearly in $|E|$.*

Proof. Let $f(n)$ denote the largest sum of cardinalities of the ground sets of leaf nodes in any decomposition tree as in Theorem 3.3.5 for a regular matroid whose ground set has cardinality n. Defining

$$\begin{aligned} g(n) := \max\Big(\{n\} &\cup \{g(t) + g(n-t) \mid 3 \leq t \leq n-3\} \\ &\cup \{g(t+1) + g(n-t+1) \mid 2 \leq t \leq n-2\} \\ &\cup \{g(t+3) + g(n-t+3) \mid 4 \leq t \leq n-4\}\Big) \end{aligned}$$

for all $n \geq 2$, and setting $g(1) := 1$, we read off from Theorem 3.3.5 that we have $f(n) \leq g(n)$ for all $n \geq 1$ (note that whenever a node with ground set of size n is the 1-, 2-, or 3-sum of its two children with ground sets of sizes n_1 and n_2, then $n_1 + n_2$ equals n minus 0, 2, or 6, respectively). Inspecting the function g more closely, we find that we have $g(7) = 15$, $g(8) = 30$ and

$$g(n) = \max\Big(\{g(t+3) + g(n-t+3) \mid 4 \leq t \leq n-4\}\Big),$$

for all $n \geq 9$. From this one deduces $g(n) = 15(n-6)$ for all $n \geq 7$ by induction. ☐

Next, for our construction it is necessary to characterize the independent sets of k-sums. We use the symbol $\uplus$ in order to emphasize when a union is taken of two sets with empty intersection.

Lemma 3.3.7. *Let $\mathcal{M} = (E, \mathcal{I})$, $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$, and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ be binary matroids with $E_1 \cap E_2 = \emptyset$ such that $\mathcal{M}$ is a k-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$. Then the independent sets of $\mathcal{M}$ can be characterized (up to isomorphism) as follows:*

- $k = 1$*:* $\mathcal{I} = \{I_1 \uplus I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\}$
- $k = 2$*: $\mathcal{M}$ is a* 1*-sum of a minor of $\mathcal{M}_1$ and a minor of $\mathcal{M}_2$; or there exist elements $r_1 \in E_1$, $r_2 \in E_2$ satisfying*

$$\begin{aligned} \mathcal{I} = \Big\{(I_1 \setminus \{r_1\}) \uplus (I_2 \setminus \{r_2\}) : I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2, \\ |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| = 1\Big\}. \end{aligned}$$

- $k = 3$: *$\mathcal{M}$ is a 2-sum of a minor of $\mathcal{M}_1$ and a minor of $\mathcal{M}_2$; or there exist pairwise distinct elements $r_1, p_1, q_1 \in E_1$, $r_2, p_2, q_2 \in E_2$ satisfying*

$$\begin{aligned}\mathcal{I} = \Big\{(I_1 \setminus \{r_1, p_1, q_1\}) \uplus (I_2 \setminus \{r_2, p_2, q_2\}) : I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2,&\\ |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| = 1,&\\ |I_1 \cap \{p_1\}| + |I_2 \cap \{p_2\}| = 1,&\\ |I_1 \cap \{q_1\}| + |I_2 \cap \{q_2\}| = 1\Big\}.&\end{aligned}$$

Proof. Note that the statement for the case $k = 1$ follows trivially from the definition of a 1-sum. Let us consider the case $k = 2$ and suppose that we have $\mathcal{M}_1 = \mathcal{M}\left(\begin{smallmatrix} a & A \end{smallmatrix}\right)$, $\mathcal{M}_2 = \mathcal{M}\left(\begin{smallmatrix} b^\intercal \\ B \end{smallmatrix}\right)$ and $\mathcal{M} = \mathcal{M}\left(\begin{smallmatrix} A & ab^\intercal \\ \mathbb{O} & B \end{smallmatrix}\right)$. The identity-extension of $\left(\begin{smallmatrix} A & ab^\intercal \\ \mathbb{O} & B \end{smallmatrix}\right)$ (after permuting columns) is

$$\begin{pmatrix} \mathbb{I} & A & \mathbb{O} & ab^\intercal \\ \mathbb{O} & \mathbb{O} & \mathbb{I} & B \end{pmatrix}.$$

Denote the elements corresponding to the first column of $\left(\begin{smallmatrix} a & A \end{smallmatrix}\right)$ and the first column of the identity-extension of $\left(\begin{smallmatrix} b^\intercal \\ B \end{smallmatrix}\right)$ (being the first unit vector) by r_1 and r_2, respectively. With this notation, we may assume that we have $E = (E_1 \setminus \{r_1\}) \uplus (E_2 \setminus \{r_2\})$. In addition, note that if $a = \mathbb{O}$ holds, then $\mathcal{M}$ is a 1-sum of $\mathcal{M}(A)$ and $\mathcal{M}(B)$, which are minors of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Thus, we may further assume that $a \neq \mathbb{O}$ holds and obtain that a subset of E is independent (in $\mathcal{M}$) if and only if it is of the form $J_1 \uplus J_2$ with $J_1 = I_1 \setminus \{r_1\}$ and $J_2 = I_2 \setminus \{r_2\}$ where $I_1 \in \mathcal{I}_1$ and (due to $a \neq \mathbb{O}$) $I_2 \in \mathcal{I}_2$ such that

$$\operatorname{span}_{J_1}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \cap \operatorname{span}_{J_2}\begin{pmatrix} \mathbb{O} & ab^\intercal \\ \mathbb{I} & B \end{pmatrix} = \{\mathbb{O}\}$$

holds, where $\operatorname{span}_J(\cdot)$ denotes the $\mathbb{F}_2$-subspace spanned by the columns corresponding to J. Because we have (with $\operatorname{span}(\cdot)$ denoting the $\mathbb{F}_2$-subspace spanned by all columns)

$$\operatorname{span}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \cap \operatorname{span}\begin{pmatrix} \mathbb{O} & ab^\intercal \\ \mathbb{I} & B \end{pmatrix} \setminus \{\mathbb{O}\} \subseteq \left\{\begin{pmatrix} a \\ \mathbb{O} \end{pmatrix}\right\},$$

the latter condition is equivalent to

$$\begin{pmatrix} a \\ \mathbb{O} \end{pmatrix} \notin \operatorname{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a \\ \mathbb{O} \end{pmatrix} \notin \operatorname{span}_{J_2} \begin{pmatrix} \mathbb{O} & ab^\intercal \\ \mathbb{I} & B \end{pmatrix},$$

which is equivalent to (recall $a \neq \mathbb{O}$)

$$a \notin \operatorname{span}_{J_1} \begin{pmatrix} \mathbb{I} & A \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ \mathbb{O} \end{pmatrix} \notin \operatorname{span}_{J_2} \begin{pmatrix} \mathbb{O}^\intercal & b^\intercal \\ \mathbb{I} & B \end{pmatrix},$$

and thus to

$$J_1 \cup \{r_1\} \in \mathcal{I}_1 \quad \text{or} \quad J_2 \cup \{r_2\} \in \mathcal{I}_2.$$

Hence, we obtain

$$\begin{aligned} \mathcal{I} &= \Big\{(I_1 \setminus \{r_1\}) \uplus (I_2 \setminus \{r_2\}) : I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2, \\ &\qquad\qquad |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| \geq 1\Big\} \\ &= \Big\{(I_1 \setminus \{r_1\}) \uplus (I_2 \setminus \{r_2\}) : I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2, \\ &\qquad\qquad |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| = 1\Big\}, \end{aligned}$$

where the last equality follows from the fact that

$$(E_1 \uplus E_2, \{I_1 \uplus I_2 \mid I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2\})$$

is an independence system (in fact, a matroid that is the direct sum of matroids).

For the remaining case $k = 3$, let $\mathcal{M}_1 = \mathcal{M}\left(\begin{smallmatrix} a & a & A \\ 0 & 1 & c^\intercal \end{smallmatrix}\right)$, $\mathcal{M}_2 = \mathcal{M}\left(\begin{smallmatrix} 0 & 1 & b^\intercal \\ d & d & B \end{smallmatrix}\right)$ and $\mathcal{M} = \mathcal{M}\left(\begin{smallmatrix} A & ab^\intercal \\ dc^\intercal & B \end{smallmatrix}\right)$. The identity extension of $\left(\begin{smallmatrix} A & ab^\intercal \\ dc^\intercal & B \end{smallmatrix}\right)$ (after permuting columns) is

$$\begin{pmatrix} \mathbb{I} & A & \mathbb{O} & ab^\intercal \\ \mathbb{O} & dc^\intercal & \mathbb{I} & B \end{pmatrix}.$$

Let us denote certain elements corresponding to the columns of the identity extensions of $\left(\begin{smallmatrix} a & a & A \\ 0 & 1 & c^\intercal \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 & b^\intercal \\ d & d & B \end{smallmatrix}\right)$, respectively, as follows:

$$\begin{array}{ccccc} \left(\begin{array}{ccccc} 1 & \mathbb{O} & a & a & A \\ \mathbb{O} & \mathbb{I} & 0 & 1 & c^\intercal \end{array}\right) \\ \begin{array}{ccccc} \uparrow & \phantom{\mathbb{I}} & \uparrow & \uparrow & \\ p_1 & & r_1 & q_1 & \end{array} \end{array} \qquad \begin{array}{c} \left(\begin{array}{ccccc} 1 & \mathbb{O} & 0 & 1 & b^\intercal \\ \mathbb{O} & \mathbb{I} & d & d & B \end{array}\right) \\ \begin{array}{ccccc} \uparrow & \phantom{\mathbb{I}} & \uparrow & \uparrow & \\ r_2 & & p_2 & q_2 & \end{array} \end{array}$$

With this notation, we may assume that we have $E = (E_1 \setminus \{r_1, p_1, q_1\}) \uplus (E_2 \setminus \{r_2, p_2, q_2\})$. In addition, note that if $d = \mathbb{O}$ holds, then $\mathcal{M}$ is a 2-sum of $\mathcal{M}\left(\begin{smallmatrix} a & A \end{smallmatrix}\right)$ and $\mathcal{M}\left(\begin{smallmatrix} b^\intercal \\ B \end{smallmatrix}\right)$, which are minors of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. A similar argument holds for the case $a = \mathbb{O}$. Thus, we may further assume that $d \neq \mathbb{O}$ and $a \neq \mathbb{O}$ holds. In this case, a subset of E is independent (in $\mathcal{M}$) if and only if it is of the form $J_1 \uplus J_2$ with $J_1 = I_1 \setminus \{r_1, p_1, q_1\}$ and $J_2 = I_2 \setminus \{r_2, p_2, q_2\}$ where (due to $d \neq \mathbb{O}$) $I_1 \in \mathcal{I}_1$ and (due to $a \neq \mathbb{O}$) $I_2 \in \mathcal{I}_2$ such that

$$\operatorname{span}_{J_1}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\intercal \end{pmatrix} \cap \operatorname{span}_{J_2}\begin{pmatrix} \mathbb{O} & ab^\intercal \\ \mathbb{I} & B \end{pmatrix} = \{\mathbb{O}\}$$

holds. Because we have

$$\operatorname{span}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\intercal \end{pmatrix} \cap \operatorname{span}\begin{pmatrix} \mathbb{O} & ab^\intercal \\ \mathbb{I} & B \end{pmatrix} \setminus \{\mathbb{O}\} \subseteq \left\{ \begin{pmatrix} a \\ \mathbb{O} \end{pmatrix}, \begin{pmatrix} \mathbb{O} \\ d \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix} \right\},$$

the latter condition is equivalent to

$$\begin{aligned}
&\left[\begin{pmatrix} a \\ \mathbb{O} \end{pmatrix} \notin \operatorname{span}_{J_1}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\intercal \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a \\ \mathbb{O} \end{pmatrix} \notin \operatorname{span}_{J_2}\begin{pmatrix} \mathbb{O} & ab^\intercal \\ \mathbb{I} & B \end{pmatrix} \right] \quad \text{and} \\
&\left[\begin{pmatrix} \mathbb{O} \\ d \end{pmatrix} \notin \operatorname{span}_{J_1}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\intercal \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mathbb{O} \\ d \end{pmatrix} \notin \operatorname{span}_{J_2}\begin{pmatrix} \mathbb{O} & ab^\intercal \\ \mathbb{I} & B \end{pmatrix} \right] \quad \text{and} \\
&\left[\begin{pmatrix} a \\ d \end{pmatrix} \notin \operatorname{span}_{J_1}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O} & dc^\intercal \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a \\ d \end{pmatrix} \notin \operatorname{span}_{J_2}\begin{pmatrix} \mathbb{O} & ab^\intercal \\ \mathbb{I} & B \end{pmatrix} \right],
\end{aligned}$$

which due to $a \neq \mathbb{O}$ and $d \neq \mathbb{O}$ is equivalent to

$$\begin{aligned}
&\left[\begin{pmatrix} a \\ 0 \end{pmatrix} \notin \operatorname{span}_{J_1}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O}^\intercal & c^\intercal \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ \mathbb{O} \end{pmatrix} \notin \operatorname{span}_{J_2}\begin{pmatrix} \mathbb{O}^\intercal & b^\intercal \\ \mathbb{I} & B \end{pmatrix} \right] \quad \text{and} \\
&\left[\begin{pmatrix} \mathbb{O} \\ 1 \end{pmatrix} \notin \operatorname{span}_{J_1}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O}^\intercal & c^\intercal \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ d \end{pmatrix} \notin \operatorname{span}_{J_2}\begin{pmatrix} \mathbb{O}^\intercal & b^\intercal \\ \mathbb{I} & B \end{pmatrix} \right] \quad \text{and} \\
&\left[\begin{pmatrix} a \\ 1 \end{pmatrix} \notin \operatorname{span}_{J_1}\begin{pmatrix} \mathbb{I} & A \\ \mathbb{O}^\intercal & c^\intercal \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ d \end{pmatrix} \notin \operatorname{span}_{J_2}\begin{pmatrix} \mathbb{O}^\intercal & b^\intercal \\ \mathbb{I} & B \end{pmatrix} \right],
\end{aligned}$$

and thus to

$$\begin{aligned}
&\left[J_1 \cup \{r_1\} \in \mathcal{I}_1 \quad \text{or} \quad J_2 \cup \{r_2\} \in \mathcal{I}_2\right] \quad \text{and}\\
&\left[J_1 \cup \{p_1\} \in \mathcal{I}_1 \quad \text{or} \quad J_2 \cup \{p_2\} \in \mathcal{I}_2\right] \quad \text{and}\\
&\left[J_1 \cup \{q_1\} \in \mathcal{I}_1 \quad \text{or} \quad J_2 \cup \{q_2\} \in \mathcal{I}_2\right].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathcal{I} = \Big\{(I_1 \setminus \{r_1, p_1, q_1\}) \uplus (I_2 \setminus \{r_2, p_2, q_2\}) :\ & I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2,\\
& |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| \geq 1,\\
& |I_1 \cap \{p_1\}| + |I_2 \cap \{p_2\}| \geq 1,\\
& |I_1 \cap \{q_1\}| + |I_2 \cap \{q_2\}| \geq 1\Big\}\\
= \Big\{(I_1 \setminus \{r_1, p_1, q_1\}) \uplus (I_2 \setminus \{r_2, p_2, q_2\}) :\ & I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2,\\
& |I_1 \cap \{r_1\}| + |I_2 \cap \{r_2\}| = 1,\\
& |I_1 \cap \{p_1\}| + |I_2 \cap \{p_2\}| = 1,\\
& |I_1 \cap \{q_1\}| + |I_2 \cap \{q_2\}| = 1\Big\},
\end{aligned}$$

where the last equality again follows from the fact that

$$(E_1 \uplus E_2, \{I_1 \uplus I_2 \mid I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\})$$

is an independence system. □

Finally, we bound the extension complexities of independence polytopes of k-sums in terms of their summands.

Lemma 3.3.8. *Let $\mathcal{M}$ be a k-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ for some $k \in \{1, 2, 3\}$. Then we have*

$$\mathrm{xc}(P(\mathcal{M})) \leq \mathrm{xc}(P(\mathcal{M}_1)) + \mathrm{xc}(P(\mathcal{M}_2)).$$

Proof. Let $\mathcal{M} = (E, \mathcal{I})$ be a k-sum of $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ (with $E_1 \cap E_2 = \emptyset$). First, observe that if some matroid $\mathcal{M}''$ is a minor of $\mathcal{M}'$, then $P(\mathcal{M}'')$ can be obtained by intersecting $P(\mathcal{M}')$ with a face of the 0/1-cube. Hence, $P(\mathcal{M}'')$ is a coordinate projection of a face of $P(\mathcal{M}')$ and therefore $\mathrm{xc}(P(\mathcal{M}'')) \leq \mathrm{xc}(P(\mathcal{M}'))$. Thus, by Lemma 3.3.7, it remains to consider the case in which there exist pairwise distinct elements $e_1, \dots, e_t \in E_1$ and pairwise distinct elements $f_1, \dots, f_t \in E_2$ such that

$$E = (E_1 \setminus \{e_1, \dots, e_t\}) \uplus (E_2 \setminus \{f_1, \dots, f_t\})$$

and

$$\begin{aligned}\mathcal{I} = \Big\{(I_1 \setminus \{e_1, \dots, e_t\}) \uplus (I_2 \setminus \{f_1, \dots, f_t\}) :\\ I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2,\\ |I_1 \cap \{e_i\}| + |I_2 \cap \{f_i\}| = 1\\ \text{for all } i \in \{1, \dots, t\}\Big\}\end{aligned}$$

holds. Thus, setting

$$Q := \Big\{(x, y) \in [0, 1]^{E_1} \times [0, 1]^{E_2} \mid x_{e_i} + y_{f_i} = 1 \ \forall\, i = 1, \dots, t\Big\},$$

we obtain that $P(\mathcal{M})$ is a coordinate projection of

$$\begin{aligned}\operatorname{conv}\Big(\big(P(\mathcal{M}_1) \times P(\mathcal{M}_2)\big) \cap Q \cap \big(\{0, 1\}^{E_1} \times \{0, 1\}^{E_2}\big)\Big)\\ = \big(P(\mathcal{M}_1) \times P(\mathcal{M}_2)\big) \cap Q,\end{aligned}$$

where the equality follows from Edmonds' intersection theorem for matroid polytopes [22] and the fact that $P(\mathcal{M}_1) \times P(\mathcal{M}_2)$ and Q are (faces of) matroid polytopes. In particular, we established

$$\begin{aligned}\operatorname{xc}(P(\mathcal{M})) &\le \operatorname{xc}\Big(\big(P(\mathcal{M}_1) \times P(\mathcal{M}_2)\big) \cap Q\Big)\\ &= \operatorname{xc}\Big(\Big\{(x, y) \in P(\mathcal{M}_1) \times P(\mathcal{M}_2) \mid x_{e_i} + y_{f_i} = 1 \ \forall\, i = 1, \dots, t\Big\}\Big)\\ &\le \operatorname{xc}\big(P(\mathcal{M}_1) \times P(\mathcal{M}_2)\big)\\ &\le \operatorname{xc}(P(\mathcal{M}_1)) + \operatorname{xc}(P(\mathcal{M}_2)).\end{aligned}$$

□

We remark that [36, Lemma 3.4] gives a similar result on the structure of independence polytopes of matroids arising from 2-sums. We are now ready to bound the extension complexities of independence polytopes of regular matroids.

Theorem 3.3.9. *For any regular matroid $\mathcal{M}$ on ground set E, we have*

$$\mathrm{xc}(P(\mathcal{M})) \leq \mathcal{O}\left(|E|^2\right).$$

Proof. Let $\mathcal{M}_1 = (E_1, \mathcal{I}_1), \ldots, \mathcal{M}_k = (E_k, \mathcal{I}_k)$ be the leaf nodes in some decomposition tree as in Theorem 3.3.5. By Lemma 3.3.8, we have that

$$\mathrm{xc}(P(\mathcal{M})) \leq \sum_{i=1}^{k} \mathrm{xc}(P(\mathcal{M}_i))$$

holds. Because there is a constant $\gamma > 0$ with $\mathrm{xc}(P(\mathcal{M}_i)) \leq \gamma \cdot |E_i|^2$ for each $i = 1, \ldots, k$ (recall that each leaf is graphic, cographic or has size bounded by 10), and $\sum_{i=1}^{k} |E_i|$ can be bounded linearly in $|E|$ due to Lemma 3.3.6, we can further estimate

$$\sum_{i=1}^{k} \mathrm{xc}(P(\mathcal{M}_i)) \leq \gamma \cdot \sum_{i=1}^{k} |E_i|^2 \leq \gamma \cdot \Big(\sum_{i=1}^{k} |E_i|\Big)^2 = \mathcal{O}(|E|^2),$$

which gives the claim. □

3.3.4 Almost-Graphic and Almost-Regular Matroids

For certain classes $\mathscr{M}$ of matroids that are closed under taking minors, there is a particular interest in the class of *almost-$\mathscr{M}$ matroids*. A matroid $\mathcal{M} = (E, \mathcal{I})$ is said to be almost-$\mathscr{M}$ if it is not contained in $\mathscr{M}$ but for every element e, the *deletion*

$$\mathcal{M} \setminus \{e\} := (E \setminus \{e\}, \{I \mid I \in \mathcal{I},\ e \notin I\})$$

is contained in $\mathscr{M}$, or the *contraction*

$$\mathcal{M}/\{e\} := (E \setminus \{e\}, \{I \setminus \{e\} \mid I \in \mathcal{I},\ e \in I\})$$

is contained in $\mathscr{M}$. Suppose we can bound the extension complexities of independence polytopes of matroids in $\mathscr{M}$. The following simple result shows that in this case we obtain a similar bound on the extension complexities of independence polytopes of almost-$\mathscr{M}$ matroids.

Proposition 3.3.10. *Let $\mathscr{M}$ be a class of matroids that is closed under taking minors and let $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function such that the extension complexity of any independence polytope of a matroid $\mathcal{M} \in \mathscr{M}$ on some ground set E is at most $f(|E|)$. Then for every almost-$\mathscr{M}$ matroid $\mathcal{M}$ on some ground set E we have that*

$$\operatorname{xc}(P(\mathcal{M})) \leq |E| \cdot f(|E| - 1) + |E| + 1$$

holds.

Proof. Let $\mathcal{M} = (E, \mathcal{I})$ be an almost-$\mathscr{M}$ matroid with $E = \{e_1, \ldots, e_n\}$ such that we have $\mathcal{M} \setminus \{e_i\} \in \mathscr{M}$ for $i = 1, \ldots, k$, and $\mathcal{M}/\{e_i\} \in \mathscr{M}$ for $i = k+1, \ldots, n$. Let us define

$$I^\star := \begin{cases} \{e_1, \ldots, e_k\} & \text{if } \{e_1, \ldots, e_k\} \in \mathcal{I}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, we have

$$\mathcal{I} = \{I^\star\} \ \cup \ \bigcup_{i=1}^{k} \{I \in \mathcal{I} \mid e_i \notin I\} \ \cup \ \bigcup_{i=k+1}^{n} \{I \in \mathcal{I} \mid e_i \in I\}.$$

For every $i = 1, \ldots, k$ let F_i denote the face of $P(\mathcal{M})$ that is defined by $x_{e_i} = 0$ and for every $i = k+1, \ldots, n$ let F_i denote the face of $P(\mathcal{M})$ that is defined by $x_{e_i} = 1$. With this notation, we obtain that

$$P(\mathcal{M}) = \operatorname{conv}\left(\{\chi(I^\star)\} \ \cup \ \bigcup_{i=1}^{n} F_i\right)$$

holds. Since the F_i's are (affinely isomorphic to) independence polytopes of matroids in $\mathscr{M}$, which are each defined on a ground set of $n-1$ elements, this implies

$$\operatorname{xc}(P(\mathcal{M})) \leq 1 + \sum_{i=1}^{n} (\operatorname{xc}(F_i) + 1) \leq 1 + \sum_{i=1}^{n} (f(n-1) + 1),$$

as claimed. □

If we replace $\mathscr{M}$ with the classes of (co-)graphic matroids and regular matroids, we obtain the classes of *almost-(co-)graphic matroids* and *almost-regular matroids*[1], respectively, see, e.g., [51], [78, Chapt. 12] or [62, Sect. 14.8.7–14.8.8]. Direct consequences of Proposition 3.3.2, Theorem 3.3.9, and Prop. 3.3.10 are the following two statements about these classes:

Corollary 3.3.11. *For any almost-(co-)graphic matroid $\mathcal{M}$ with ground set E we have*

$$\mathrm{xc}(P(\mathcal{M})) \le \mathcal{O}(|E|^3).$$

Corollary 3.3.12. *For any almost-regular matroid $\mathcal{M}$ with ground set E we have*

$$\mathrm{xc}(P(\mathcal{M})) \le \mathcal{O}(|E|^3).$$

3.4 Cut Dominant

In Section 3.3, we have seen how polynomial-size extended formulations for the spanning-tree polytope can be used to bound the extension complexities of independence polytopes of certain classes of matroids (that are closely related to graphic matroids). This may not be surprising since the spanning-tree polytope itself is a face of an independence polytope of a (graphic) matroid. In this section, however, we show that any extended formulation for $P_{\text{sp.trees}}(G)$ can be transferred to one for another well-known polyhedron, which – at a first sight – does not seem to be closely related to $P_{\text{sp.trees}}(G)$.

Let $G = (V, E)$ be a connected undirected graph. Given a set of nodes $S \subseteq V$, the *cut* $\delta(S)$ of G is the set of all edges in E having exactly one endnode in S. Given nonnegative weights $c \in \mathbb{R}^E_{\ge 0}$, the classic *minimum-cut* problem asks for finding a set S with $\emptyset \subsetneq S \subsetneq V$) for which $\sum_{e\in\delta(S)} c_e$ is smallest possible. A polyhedron that is naturally associated to this problem is the *cut dominant* of G, defined via

$$P_{\text{cut+}}(G) := \mathrm{conv}\left(\left\{\chi(\delta(S)) \in \{0,1\}^E \mid \emptyset \subsetneq S \subsetneq V\right\}\right) + \mathbb{R}^E_{\ge 0}.$$

For interesting properties of $P_{\text{cut+}}(G)$, we refer to [16]. It is easy to check that $P_{\text{cut+}}(G)$ is the (inclusion-wise) largest polyhedron P such that for

[1] The definition of almost-regular matroids in [78] slightly differs from the one described here, i.e., it describes a subclass of our notion of almost-regular matroids (which is identical to the one in [51]). It turns out that Corollary 3.3.12 is valid w.r.t. both definitions.

every $c \in \mathbb{R}^E_{\geq 0}$, $\min \{\langle c, x\rangle \mid x \in P\}$ is equal to the smallest weight (with respect to c) of any nonempty cut in G. Because the minimum-cut problem is polynomially-time solvable, optimizing a linear function over $P_{\text{cut+}}(G)$ can be done in polynomial time as well. However, a complete characterization of the facets of $P_{\text{cut+}}(G)$ is not known. In contrast, while the number of its facets grows exponentially in the dimension, $P_{\text{cut+}}(G)$ can be described by extended formulations of size $\mathcal{O}(|V| \cdot |E|)$, see [77] or [14]. Here, we provide an alternative upper bound on the extension complexity of $P_{\text{cut+}}(G)$, involving the extension complexity of $P_{\text{sp.trees}}(G)$:

Theorem 3.4.1. *Let $G = (V, E)$ be a connected undirected graph. Then we have*

$$\operatorname{xc}(P_{\text{cut+}}(G)) \leq \operatorname{xc}(P_{\text{sp.trees}}(G)) + \mathcal{O}(|E|).$$

Proof. Let $D = (V, A)$ be the directed graph arising from G by bidirecting each edge. Given a set of nodes $S \subseteq V$, define $\vec{\delta}(S)$ as the set of arcs $(v, w) \in A$ with $v \in S$ and $w \notin S$. Let us fix some node $r \in V$ and consider the polyhedron

$$P_{\text{r-cutdom}}(D) := \operatorname{conv}\left(\left\{\chi(\vec{\delta}(S)) \in \{0,1\}^A \mid r \in S \subsetneq V\right\}\right) + \mathbb{R}^A_{\geq 0}.$$

It is easy to check that $P_{\text{cut+}}(G)$ is the image of $P_{\text{r-cutdom}}(D)$ under the linear map $\pi \colon \mathbb{R}^A \to \mathbb{R}^E$ defined via $\pi(y)_{\{v,w\}} := y_{(v,w)} + y_{(w,v)}$, see, e.g. [19]. By a classic result of Edmonds [24], $P_{\text{r-cutdom}}(D)$ can be described via

$$P_{\text{r-cutdom}}(D) = \left\{ y \in \mathbb{R}^A_{\geq 0} \mid \sum_{a \in B} y_a \geq 1 \text{ for all } r\text{-arborescences } B \subseteq A \right\},$$

where a set $B \subseteq A$ is called an *r-arborescence* if for any node $v \in V \setminus \{r\}$ there exists exactly one directed path from r to v in B. Let us define the polytope

$$P_{\text{r-arb}}(D) := \operatorname{conv}\left\{\chi(B) \in \{0,1\}^A \mid B \subseteq A \ r\text{-arborescence}\right\},$$

and observe that we have

$$P_{\text{r-cutdom}}(D) = \left\{ y \in \mathbb{R}^A_{\geq 0} \mid \langle y, z\rangle \geq 1 \text{ for all } z \in P_{\text{r-arb}}(D)\right\}.$$

Thus, by Proposition 3.1.2, we obtain

$$\mathrm{xc}(P_{\mathrm{cut+}}(G)) \leq \mathrm{xc}(P_{\mathrm{r-cutdom}}(D)) \leq \mathrm{xc}(P_{\mathrm{r-arb}}(D)) + \mathcal{O}(|E|). \tag{3.14}$$

Consider the two sets

$$\begin{aligned} M_1 &:= \Big\{ B \subseteq A \mid |\{(v,w),\,(w,v)\} \cap B| \leq 1 \text{ for all } \{v,w\} \in E, \\ &\qquad\qquad \Big\{\{v,w\} \mid (v,w) \in B\Big\} \text{ is a spanning tree in } G \Big\}, \\ M_2 &:= \Big\{ B \subseteq A \mid |\{(w,v) \mid (w,v) \in B\}| = 1 \text{ for all } v \in V \setminus \{r\} \Big\}. \end{aligned}$$

Note that $M_1 \cap M_2$ equals the set of r-arborescences, and hence

$$P_{\mathrm{r-arb}}(D) = \mathrm{conv}\,\{\chi(B) \mid B \in M_1 \cap M_2\}\,.$$

Furthermore, it is easy to check that M_1 and M_2 describe bases of certain matroids, respectively. Setting $Q_i := \mathrm{conv}\,\{\chi(B) \mid B \in M_i\}$ for $i = 1, 2$, by Edmonds' intersection theorem for matroid polytopes [22] we thus obtain $Q = Q_1 \cap Q_2$ and hence

$$\mathrm{xc}(P_{\mathrm{r-arb}}(D)) \leq \mathrm{xc}(Q_1) + \mathrm{xc}(Q_2). \tag{3.15}$$

Now note that we have

$$Q_1 = \Big\{ z \in [0,1]^A \mid x_{\{v,w\}} = z_{(v,w)} + z_{(w,v)} \text{ for all } \{v,w\} \in E,\ x \in P_{\mathrm{sp.trees}}(G) \Big\},$$

as well as

$$Q_2 = \Big\{ z \in [0,1]^A \mid \sum_{(v,w) \in A} z_{(v,w)} = 1 \text{ for all } v \in V \setminus r \Big\}.$$

Thus, we obtain $\mathrm{xc}(Q_1) \leq \mathrm{xc}(P_{\mathrm{sp.trees}}(G)) + \mathcal{O}(|E|)$ and $\mathrm{xc}(Q_2) \leq \mathcal{O}(|E|)$, and hence the claim follows by Inequality (3.14) and Inequality (3.15).

□

As a simple consequence, Theorem 3.4.1 together with Inequality (3.6) implies the known bound $\mathrm{xc}(P_{\mathrm{cut+}}(G)) \leq \mathcal{O}(|V| \cdot |E|)$ for general graphs. Another consequence, for which we are not aware of any reference, is the following:

Corollary 3.4.2. *Let $G = (V, E)$ be a connected undirected planar graph. Then we have* $\operatorname{xc}(P_{\text{cut+}}(G)) \leq \mathcal{O}(|E|)$.

Proof. Follows from Theorem 3.4.1 and Williams' linear-size extended formulation for $P_{\text{sp.trees}}(G)$ for planar graphs [79]. □

4

Describing Polytopes: Obstructions

4.1 General Lower Bounds

Extended formulations of small size have already been investigated since the early days of combinatorial optimization. In contrast, the search for general lower bounds on sizes of such formulations is a relatively young research field. In 1991, Yannakakis [80] gave an important algebraic interpretation of the extension complexity that later served as the basis for most results concerning lower bounds on sizes of extended formulations: Given a polytope $P \subseteq \mathbb{R}^d$, a nonnegative matrix $S \in \mathbb{R}^{m \times n}_{\geq 0}$ is called a *slack matrix* of P if there exist a set of points $v_1, \ldots, v_n \in \mathbb{R}^d$, a system of m linear inequalities $Ax \leq b$, and a (possibly empty) system of linear equations $Cx = d$ with

$$P = \operatorname{conv}(\{v_1, \ldots, v_n\}) = \left\{x \in \mathbb{R}^d \mid Ax \leq b,\, Cx = d\right\},$$

such that

$$S_{i,j} = b_i - A_{i,*} \cdot v_j$$

holds for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, where $A_{i,*}$ denotes the i'th row of A. It turns out that the extension complexity of a polytope can be expressed in terms of a property of S that is similar to the classical rank.

Theorem 4.1.1 ([80]). *Let P be a polytope with* $\dim(P) \geq 1$ *and S be any of its slack matrices. Then the extension complexity of P equals the* nonnegative rank *of S, i.e., the smallest number r such that there exist nonnegative matrices $U \in \mathbb{R}^{m \times r}_{\geq 0}$, $V \in \mathbb{R}^{r \times n}_{\geq 0}$ with $S = U \cdot V$.*

In his work, Yannakakis already mentioned existing lower bounds on the nonnegative rank of a matrix. However, his ideas only led to applications in the last few years. In the following sections, we review prominent general lower bounds on the extension complexity that are based on Theorem 4.1.1 and shed light on some issues concerning these bounds.

4.1.1 Rectangle Coverings

An important bound on the extension complexity that only depends on the combinatorial structure of a polytope, is the *rectangle-covering* bound. In the context of extended formulations, this bound has been studied first in [30] and was later used in [29] to establish a super-polynomial lower bound on the extension complexity of the traveling-salesman polytope.

Let $S \in \mathbb{R}^{m \times n}_{\geq 0}$ be any nonnegative matrix. We say that a set $R = A \times B$ with $A \subseteq \{1, \dots, m\}$ and $B \subseteq \{1, \dots, n\}$ is a *rectangle* of S if we have $S_{i,j} > 0$ for all $(i, j) \in R$. A *rectangle covering* of S is a set of rectangles $R_1, \dots, R_k$ of S such that every entry (i, j) with $S_{i,j} > 0$ is contained in at least one of the R_ℓ's. Let rec-cov(S) be the smallest number of rectangles in any rectangle covering of S. If we restrict S to be a slack matrix of some polytope P, it can be shown that the value of rec-cov(P) := rec-cov(S) does not depend on the actual choice of S. Furthermore, we have the following, very useful relation:

Proposition 4.1.2 (see, e.g., [30]). *For every polytope P we have that* rec-cov$(P) \leq$ xc(P) *holds.*

The value of rec-cov(P) is called the *rectangle-covering number* of P. For some polytopes, the rectangle-covering number provides a tight bound on their extension complexity. This is the case, for instance, for combinatorial cubes, simplices, and (two-level) transportation polytopes [30]. For some polytopes associated to prominent combinatorial-optimization problems, the rectangle-covering number can be shown to grow at least superpolynomially in their dimension. This has been the crucial ingredient for answering questions concerning the existence of polynomial-size extended formulations, e.g., for the traveling-salesman polytope, see [29]. In some cases, however, the gap between the extension complexity and rectangle-covering number of a polytope can be arbitrarily large. This turns out to be the case for the matching-polytope. While Yannakakis [80] observed that the rectangle-covering number of $P_{\mathrm{match}}(n)$ can be bounded by a polynomial in n, Rothvoß [72] showed that its extension complexity grows exponentially in n.

We show that the same is true for independence polytopes of certain matroids. As already mentioned, Rothvoß' [71] counting arguments yield the existence of a family of independence polytopes of matroids $(\mathcal{M}_n)_{n=1,2,\dots}$ such that each matroid $\mathcal{M}_n$ is defined on a ground set of n elements and for which $\operatorname{xc}(P(\mathcal{M}_n))$ grows exponentially in n. Unfortunately, no such family is known explicitly. However, even if there were candidate polytopes, the rectangle-covering number would not be strong enough to prove an exponential growth of their extension complexities:

Proposition 4.1.3. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then we have*

$$\text{rec-cov}(P(\mathcal{M})) \leq \mathcal{O}(|E|^2).$$

Proof. Let $r\colon 2^E \to \mathbb{Z}_{\geq 0}$ denote the rank function of $\mathcal{M}$. It is well-known [22] that $P(\mathcal{M})$ equals the set of points $x \in \mathbb{R}^E$ satisfying

$$x_e \geq 0 \quad \text{for all } e \in E, \tag{4.1}$$

$$\sum_{e \in F} x_e \leq r(F) \quad \text{for all } F \subseteq E. \tag{4.2}$$

Let S be a slack matrix of $P(\mathcal{M})$ whose columns are indexed by the vertices of $P(\mathcal{M})$ (i.e., characteristic vectors of independent sets of $\mathcal{M}$) and whose rows correspond to the above inequalities. Furthermore, let S' be the submatrix of S only consisting of rows corresponding to the inequalities in (4.2). Since the submatrix of S that only consists of rows corresponding to the inequalities in (4.1) has only $|E|$ rows and hence can be covered by $|E|$ many rectangles, we have $\text{rec-cov}(S) \leq \text{rec-cov}(S') + |E|$. Thus, it remains to show that we have $\text{rec-cov}(S') \leq \mathcal{O}(|E|^2)$.

To this end, let us index the rows of S' by all sets $F \subseteq E$ and the columns of S' by all independent sets $I \in \mathcal{I}$. Note that an entry $S'_{F,I}$ is positive if and only if $|I \cap F| \leq r(F) - 1$ holds. Note that this is the case if and only if

- there exists some $f \in F \setminus I$ such that $I \cup \{f\} \in \mathcal{I}$, or
- there exist some $f \in F \setminus I$ and $e \in I \setminus F$ such that $(I \setminus \{e\}) \cup \{f\} \in \mathcal{I}$.

For the necessity of the latter claim, first observe that there exists an element $f \in F \setminus I$ such that $(I \cap F) \cup \{f\} \in \mathcal{I}$. Second, note that we have $r(I \cup \{f\}) \geq |I|$. Thus, if we have $I \cup \{f\} \notin \mathcal{I}$, then we can complete $(I \cap F) \cup \{f\}$ with elements in I to an independent set of cardinality at least $|I|$. Hence there exists an element $g \in I \setminus F$ such that $(I \setminus \{g\}) \cup \{f\} \in \mathcal{I}$ holds.

Thus, each non-incidence is contained in a rectangle of type

$$\{F \subseteq E \mid f \in F\} \times \{I \in \mathcal{I} \mid f \notin I, I \cup \{f\} \in \mathcal{I}\}$$

for some $f \in E$, or in a rectangle of type

$$\{F \subseteq E \mid f \in F, g \notin F\} \times \{I \in \mathcal{I} \mid f \notin I, g \in I, (I \setminus \{g\}) \cup \{f\} \in \mathcal{I}\}$$

for some $f, g \in E$. Thus we can cover the positive entries in S' with at most $|E|^2$ rectangles. □

4.1.2 Fractional Rectangle Coverings

Instead of determining the rectangle-covering number of a polytope P – which is a difficult task – one usually computes lower bounds on rec-cov(P) that are (potentially) simpler to evaluate. Most of these bounds are direct translations of known bounds on the chromatic number of a graph when interpreting the problem of finding a smallest rectangle covering as a graph-coloring problem, see, e.g. [65, Sec. 4.15].

One of these lower bounds is based on the concept of cliques in a graph: Let S be a slack matrix of some polytope P and let $\mathcal{F}$ be a set of entries (i, j) with $S_{i,j} > 0$. Such a set $\mathcal{F}$ is called a *fooling set* if we have for every two entries $(i_1, j_1), (i_2, j_2) \in \mathcal{F}$ that $S_{i_1,j_2} = 0$ or $S_{i_2,j_1} = 0$ holds. In this case, because a rectangle is only allowed to cover positive entries, any rectangle of S covers at most one entry of $\mathcal{F}$. Thus, every rectangle covering needs at least $|\mathcal{F}|$ rectangles. For certain simple examples (such as simplices or cubes), it can be shown that there exist fooling sets that have cardinality $\mathrm{xc}(P)$ and hence provide tight bounds on the extension complexity. However, it is known that every fooling set has cardinality of at most $(\dim(P) + 1)^2$, see [30, Lem. 5.7]. Thus, this method is not applicable for providing exponential lower bounds on sizes of extended formulations.

Many results concerning (combinatorial) exponential lower bounds on the extension complexity rely on an approach that can be regarded as a generalization of the previous idea. Given a set $\mathcal{T}$ of entries (i, j) with $S_{i,j} > 0$, let $\alpha(\mathcal{T})$ denote the largest number of entries of $\mathcal{T}$ that can be covered by any rectangle of S. Clearly, we have that $\text{rec-cov}(S) \geq \frac{|\mathcal{T}|}{\alpha(\mathcal{T})}$ holds. It turns out that for many polytopes P that are associated to hard combinatorial-optimization problems, one can find such a set $\mathcal{T}$ with the property that the previous bound grows exponentially in the dimension of P. We will revisit this approach in Section 4.2. For more information on the above bounds we refer to [30].

In this section, we consider another bound on the rectangle-covering number, which is closely related to the fractional chromatic number of a graph. Given a slack matrix S of some polytope P, let $\mathcal{R}$ denote the set of all rectangles of S. A *fractional rectangle-covering* of S is a nonnegative vector $\lambda \in \mathbb{R}^{\mathcal{R}}_{\geq 0}$ such that

$$\sum_{\substack{R \in \mathcal{R}, \\ (i,j) \in R}} \lambda_R \geq 1$$

holds for all entries (i, j) with $S_{i,j} > 0$. The *size* of a fractional rectangle-covering is defined as the sum of all entries of λ. Let $\text{frec-cov}(S)$ denote the smallest size of any fractional rectangle-covering of S. Similar to the case of rectangle coverings, it can be shown that the value of $\text{frec-cov}(S)$ does not depend on the actual choice of S. Thus, we may define $\text{frec-cov}(P) := \text{frec-cov}(S)$ to be the *fractional rectangle-covering number* of P. Since rectangle coverings correspond to fractional rectangle-coverings whose entries have only values in $\{0, 1\}$, we clearly have that

$$\text{frec-cov}(P) \leq \text{rec-cov}(P)$$

holds. On the other side, it is easy to check (and well-known in the context of (fractional) graph colorings) that the fractional rectangle-covering number dominates all previously defined lower bounds on the rectangle covering number.

Let K_n denote the complete graph on n nodes. In Section 3.2 we have seen that the extension complexity of the spanning-tree polytope $P_{\text{sp.trees}}(K_n)$ can be bounded from above by $O(n^3)$. It is an open question whether this bound can be improved. In particular, the same question is open for the rectangle-covering number of $P_{\text{sp.trees}}(K_n)$. To our knowledge, the best known lower bound on both numbers is $\Omega(n^2)$, which follows from basic arguments involving the dimension of $P_{\text{sp.trees}}(K_n)$. However, we believe that the $O(n^3)$-bound is tight. Unfortunately, we will show that the lower bounds on $\text{rec-cov}(P_{\text{sp.trees}}(K_n))$ presented in this section cannot confirm this conjecture:

Proposition 4.1.4. *The fractional rectangle-covering number of the spanning-tree polytope of K_n is at most $O(n^{2+2/3} \log n)$.*

In the proof of this statement we make use of the following lemma, which can be regarded as a special case of Lemma 3.3 in [30].

Lemma 4.1.5 (see Lemma 3.3 in [30] for $\ell = 1$)**.** *Let V be some finite set and $k \geq 1$ any integer. Then there exist sets $C_1, \dots, C_t \subseteq V$ with $t \leq O(k^2 \log |V|)$ with the following property: For every $U \subseteq V$ with $|U| \leq k$ and every $v \in V \setminus U$ there exists some i with $U \subseteq C_i$ and $v \notin C_i$.*

Proof. Assume $|V| \geq 2$ and define $\mathcal{D} := \{(U, v) \mid U \subseteq V, |U| \leq k, v \in V \setminus U\}$. We say that a set $C \subseteq V$ *separates* $(U, v) \in \mathcal{D}$ if $U \subseteq C$ and $v \notin C$. Suppose we pick a set $C \subseteq V$ by selecting the elements from V independently with probablity $p := \frac{k}{k+1}$. For every fixed pair $(U, v) \in \mathcal{D}$, the probability that C separates (U, v) is at least

$$p^k(1-p) = \left(\frac{k}{k+1}\right)^k \cdot \frac{1}{k+1} = \left(1 - \frac{1}{k+1}\right)^{k+1} \cdot \frac{1}{k} \geq \frac{1}{ck},$$

where $c > 1$ is some constant (recall that $(1 - \frac{1}{k+1})^{k+1} > 0$ tends to $\frac{1}{e}$ as k grows). Suppose now we choose independently t such sets C and denote them by $C_1, \dots, C_t$. As we have

$$|\mathcal{D}| \leq \{U \subseteq V : |U| \leq k\}| \cdot |V| \leq |V|^{k+1} \cdot |V|,$$

the expected number of pairs in $\mathcal{D}$ that are not separated by any of the C_i's is at most

$$|V|^{k+2} \cdot (1 - \tfrac{1}{ck})^t. \tag{4.3}$$

The latter term is strictly less than 1 if we choose t strictly larger than

$$-(k+2)\frac{\log|V|}{\log(1-\frac{1}{ck})} = (k+2)\frac{\log|V|}{\log(1+\frac{1}{ck-1})} = (k+2)\frac{\log|V|}{\frac{1}{ck-1}\log((1+\frac{1}{ck-1})^{ck-1})},$$

which can be bounded from above by $c' \cdot k^2 \log|V|$ for some constant $c' > 0$ (since $(1+\frac{1}{ck-1})^{ck-1} > 0$ converges to e as k grows). Thus, choosing t as described this ensures the existence of sets $C_1, \dots, C_r \subseteq V$ such that every pair in $\mathcal{D}$ is separated by at least one of the C_i's. □

Note that whenever we have $U \subseteq C_i$ and $v \notin C_i$, the set C_i can be regarded as a certificate that v is not contained in U. The above statement asserts that we do not need many of such sets in order to certify non-containment in every case where the cardinality of U is bounded by k.

Proof of Proposition 4.1.4. Recall that the spanning-tree polytope of $K_n = (V, E)$ consists of all points $x \in \mathbb{R}^E$ satisfying

$$x_e \geq 0 \quad \text{for all } e \in E, \tag{4.4}$$

$$x(E(U)) \leq |U| - 1 \quad \text{for all } \emptyset \neq U \subseteq V, \tag{4.5}$$

as well as $x(E) = |V| - 1$. Let S be a slack matrix of $P_{\text{sp.trees}}(K_n)$ whose columns are indexed by the vertices of $P_{\text{sp.trees}}(K_n)$ (i.e., characteristic vectors of spanning trees in K_n) and whose rows correspond to the above inequalities. We partition S into three submatrices and show that each of them admits a fractional rectangle-covering of the desired size. Let S' be the submatrix of S only consisting of rows corresponding to the inequalities in (4.4). Fixing any number $k \geq 0$, let S'' denote the submatrix of S whose rows correspond to the inequalities in (4.5) with $|U| \leq k$. The submatrix of S that is formed by the remaining rows, which correspond to the inequalities in (4.5) with $|U| > k$, is denoted by S'''.

First, observe that S' has only $|E|$ rows and hence

$$\text{frec-cov}(S') \leq \text{rec-cov}(S') \leq |E| = \mathcal{O}(n^2) \tag{4.6}$$

holds. Second, let us index the rows of S'' by all sets $\emptyset \neq U \subseteq V$ with $|U| \leq k$ and the columns of S'' by all spanning trees $T \subseteq E$. Note that an entry $S''_{U,T}$ is positive if and only if the number of connected components of $T \cap E(U)$ is at least two. For any spanning tree $T \subseteq E$ and nodes $u, w \in V$ let $\operatorname{int}(T, u, w)$ denote the set of nodes on the (unique) path from u to w in T that are distinct from u and w. Observe that the number of connected components of $T \cap E(U)$ is at least two if and only if there exist three nodes $u, v, w \in V$ with $u, w \in U$ and $v \in \operatorname{int}(T, u, w) \setminus U$. Thus, choosing $C_1, \ldots, C_t$ as in Lemma 4.1.5, we have that every entry (U, T) with $S''_{U,T} > 0$ is covered by some rectangle

$$\begin{aligned}\{U \subseteq V \mid U \neq \emptyset, |U| \leq k, u, w \in U, U \subseteq C_i\} \\ \times \{T \subseteq E \mid T \text{ spanning tree}, \exists v \in \operatorname{int}(T, u, w) : v \notin C_i\}\end{aligned}$$

for a certain choice of nodes $u, w \in V$ and some index $i \in \{1, \ldots, t\}$. Since the number of such rectangles is $\mathcal{O}(|V|^2 \cdot t)$ and we may assume $t \leq \mathcal{O}(k^2 \log n)$ by Lemma 4.1.5, we thus obtained that

$$\text{frec-cov}(S'') \leq \text{rec-cov}(S'') \leq \mathcal{O}(n^2 \cdot k^2 \log n) \tag{4.7}$$

holds.

Finally, consider S''' and index its rows by all sets $\emptyset \neq U \subseteq V$ with $|U| > k$ and its columns by all spanning trees $T \subseteq E$. By the previous observations, we have that every entry (U, T) with $S'''_{U,T} > 0$ is covered by some rectangle

$$\begin{aligned}R_{u,v,w} := \{U \subseteq V \mid U \neq \emptyset, |U| > k, u, w \in U, v \notin U\} \\ \times \{T \subseteq E \mid T \text{ spanning tree}, v \in \operatorname{int}(T, u, w)\}\end{aligned} \tag{4.8}$$

for a certain choice of nodes $u, v, w \in V$. We claim that the number of such rectangles covering an entry (U, T) with $S'''_{U,T} > 0$ is at least k: Let U_1 denote one connected component of the graph $(U, T \cap E(U))$. Then for every two nodes $u \in U_1$, $w \in U \setminus U_1$ (recall that $U \setminus U_1$ is nonempty due to $S'''_{U,T} > 0$), there exists some node $v \in V \setminus U$ with $v \in \operatorname{int}(T, v, w)$. The number of rectangles of the above type that cover (U, T) is hence indeed at least

$$|U_1| \cdot (|U| - |U_1|) \geq 1 \cdot (|U| - 1) \geq k.$$

Thus, we obtain a fractional rectangle-covering λ of S''' by setting $\lambda_R := \frac{1}{k}$ if R is of type (4.8), and $\lambda_R := 0$ else. This implies

$$\text{frec-cov}(S''') \leq |V|^3 \cdot \frac{1}{k} = \frac{n^3}{k}. \tag{4.9}$$

Choosing $k := \lfloor n^{1/3} \rfloor$, with the inequalities in (4.6), (4.7), and (4.9), we can summarize

$$\begin{aligned} \text{frec-cov}(S) &\leq \text{frec-cov}(S') + \text{frec-cov}(S'') + \text{frec-cov}(S''') \\ &\leq \mathcal{O}(n^2 + n^2 \cdot k^2 \log n + \tfrac{n^3}{k}) \\ &= \mathcal{O}(n^2 + n^{2+2/3} \log n + n^{2+2/3}) = \mathcal{O}(n^{2+2/3} \log n). \end{aligned}$$

□

4.1.3 Hyperplane-Separation Bound

The rectangle-covering number is a bound on the nonnegative rank of a matrix S that only depends on the sparsity pattern of S. As a consequence, for some polytopes the rectangle-covering number differs from the extension complexity by orders of magnitude. In the case of the matching-polytope, this issue was resolved by considering the *hyperplane-separation* bound. Let P be a polytope with slack matrix $S \in \mathbb{R}^{m \times n}_{\geq 0}$ and let $W \in \mathbb{R}^{m \times n}$ be a set of weights on the entries of S. For every rectangle R of S, let us use the notation $\langle W, R \rangle := \sum_{(i,j) \in R} W_{i,j}$. Furthermore, let $\|S\|_\infty$ denote the largest entry of S and set $\langle W, S \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} W_{i,j} S_{i,j}$. Based on Theorem 4.1.1, it can be shown that the expression

$$H(S, W) := \frac{\langle W, S \rangle}{\|S\|_\infty \cdot \max\{\langle W, R \rangle \mid R \text{ rectangle of } S\}} \tag{4.10}$$

is a lower bound on the nonnegative rank of S and hence on the extension complexity of P, see [72, Lem. 1]. As Rothvoß showed, this technique is very useful to prove strong lower bounds on sizes of classical extended formulations.

In this section, we show that, in some situations, it can be also applied to obtain bounds on sizes of *approximate* extended formulations. Given a polytope P containing the origin and a factor $\varrho \geq 1$, we say that a polyhedron Q together with some affine map π is a ϱ-*approximate extension* if $P \subseteq \pi(Q) \subseteq \varrho P$. Observe that maximizing a linear function c over the image of a ϱ-approximate extension yields an optimal value that can be bounded by $\varrho \cdot \max\{\langle c, x\rangle \mid x \in P\}$. Clearly, for large enough ϱ, ϱ-approximate extensions can have much fewer facets than classical extensions.

Lower bounds on sizes of ϱ-approximate extensions have been first studied in [10]. For a polytope $P \subseteq \mathbb{R}^d$ containing the origin, a non-negative matrix $S \in \mathbb{R}^{m\times n}_{\geq 0}$ is called a *ϱ-approximate slack matrix* of P if there exists a set of points $v_1, \dots, v_n \in \mathbb{R}^d$, a system of m linear inequalities $Ax \leq b$, and a (possibly empty) system of linear equations $Cx = d$ with

$$P = \operatorname{conv}(\{v_1, \dots, v_n\}) = \left\{x \in \mathbb{R}^d \mid Ax \leq b,\ Cx = d\right\},$$

such that

$$S_{i,j} = \varrho b_i - A_{i,*} \cdot v_j$$

holds for all $i = 1, \dots, m$ and $j = 1, \dots, n$, where $A_{i,*}$ denotes the i'th row of A. The following result, which was used in [10] and proved[1] in [65, Lem. 4.1 & Lem. 4.2], generalizes Theorem 4.1.1:

Theorem 4.1.6. *Let P be a polytope containing the origin and* $\dim(P) \geq 1$, *$\varrho \geq 1$, and S a ϱ-approximate slack matrix of P. Then the smallest size of any ϱ-approximate extension of P equals the nonnegative rank of S.*

Using the hyperplane-separation bound, let us give a simple lower bound on the extension complexity of a square matrix.

Lemma 4.1.7. *Let $S \in \mathbb{R}^{m\times m}_{\geq 0}$ be a nonnegative square matrix with $S \neq \mathbb{O}$. Setting $\alpha := \min_{1\leq i\leq n} S_{i,i}$ and $\beta := \max_{1\leq i<j\leq m}(\min\{S_{i,j}, S_{j,i}\})$, we have that the nonnegative rank of S is at least $\frac{\alpha-\beta\cdot m}{\|S\|_\infty}\cdot m$.*

[1] For general polyhedra, Theorem 4.1.6 is only valid up to an additive term of at most one. Here, we used the fact that if P is a polytope with $\dim(P) \geq 1$ and $Ax \leq b$ any system of linear inequalities describing P, then the inequalitiy $0 \leq 1$ is a nonnegative combination from inequalities in $Ax \leq b$. See the proof in [65, Lem. 4.2] for details.

Proof. Let us define the weights W as follows:

$$W_{i,j} := \begin{cases} 1 & \text{if } i = j, \\ -2 & \text{if } S_{i,j} < S_{j,i}, \\ -2 & \text{if } S_{i,j} = S_{j,i} \text{ and } i < j, \\ 0 & \text{else.} \end{cases}$$

For any rectangle R of S that contains k diagonal entries, we have

$$\langle W, R\rangle \leq k - 2 \cdot \tfrac{k(k-1)}{2} = 1 - (k-1)^2 \leq 1,$$

with equality if R consists of exactly one diagonal entry. Thus, evaluating the expression in (4.10), we obtain that the nonnegative rank of S is at least

$$H(S, W) = \frac{\langle W, S\rangle}{\|S\|_\infty \cdot 1}.$$

By the definitions of α and β, we have

$$\langle W, S\rangle \geq \alpha \cdot m - 2 \cdot \beta \cdot \tfrac{m(m-1)}{2} \geq \alpha \cdot m - \beta \cdot m^2,$$

which gives the claim. □

Note that if the diagonal entries of S are a fooling set, then we simply have $\beta = 0$ in the above statement. Thus, in this case the bound of Lemma 4.1.7 is only as good as the combinatorial fooling-set bound if α coincides with the largest entry of S. Nevertheless, the bound of Lemma 4.1.7 still yields useful bounds for small perturbations of S, which may completely destroy the sparsity pattern of S.

In [8], Braun & Pokutta provide another, information-theoretic bound on the nonnegative rank of matrices having an "approximate fooling set". As an application, it is shown that every $(1 + \varepsilon)$-approximate extension of the cube $[0, 1]^n$ with $\varepsilon = \frac{1}{16n^2}$ has at least $\sqrt{2} \cdot n$ facets, see [8, Cor. 6.4]. Using the simple bound of Lemma 4.1.7, we can improve upon their result.

Proposition 4.1.8. *Every* $(1 + \varepsilon)$*-approximation of* $[0, 1]^n$ *has at least* $(1 - 2\varepsilon n) \cdot 2n$ *many facets.*

Proof. Let S be the slack matrix of $[0,1]^n$ whose columns are indexed by all vertices of $[0,1]^n$ and whose rows correspond to the inequalities $x_i \geq 0$ and $x_i \leq 1$ for $i = 1, \ldots, n$. Note that all entries of S are in $\{0,1\}$. Let $\tilde{S}$ denote the $1+\varepsilon$-approximate slack matrix of $[0,1]^n$ with columns and rows in the same order as in S. In [30, Prop. 5.9] it was shown that the matrix S has a fooling set of size $2n$. Thus, up to permutation of columns and rows, S contains a submatrix $S' \in \{0,1\}^{2n \times 2n}$ with a fooling set of size $2n$ on its diagonal. Let $\tilde{S}' \in \mathbb{R}^{2n}$ be the corresponding submatrix of $\tilde{S}$.

Observe first that all entries of $\tilde{S}'$ have values in $\{0, \varepsilon, 1+\varepsilon\}$. Second, all diagonal entries of $\tilde{S}'$ have value $1+\varepsilon$. Third, for every i, j with $i \neq j$, we have that $\tilde{S}'_{i,j} \in \{0, \varepsilon\}$ or $\tilde{S}'_{j,i} \in \{0, \varepsilon\}$ holds. Thus, setting $S = \tilde{S}'$ and defining α and β as in Lemma 4.1.7, we have $\alpha = 1+\varepsilon$ and $\beta \leq \varepsilon$. Now Lemma 4.1.7 states that the nonnegative rank of $\tilde{S}'$ (and hence also the nonnegative rank of $\tilde{S}$) is at least

$$\frac{(1+\varepsilon) - \varepsilon \cdot 2n}{1+\varepsilon} \cdot 2n = (1 - \tfrac{2\varepsilon n}{1+\varepsilon}) \cdot 2n \geq (1 - 2\varepsilon n) \cdot 2n.$$

The claim now follows from Theorem 4.1.6. □

In particular, this means that every $(1 + \frac{1}{16n^2})$-approximation of $[0,1]^n$ still has at least $\lceil 2n - \frac{1}{4} \rceil = 2n$ many facets.

4.2 Correlation Polytope

In this section, we provide a simple and short proof for a superpolynomial lower bound on the extension complexity of the *correlation polytope*

$$P_{\mathrm{corr}}(n) := \operatorname{conv}\{bb^\intercal \mid b \in \{0,1\}^n\}.$$

This polytope has been the first explicit example of a 0/1-polytope whose extension complexity is not bounded by a polynomial in its dimension, which was first shown by Fiorini et al. [29]. More precisely, they showed that $\mathrm{xc}(P_{\mathrm{corr}}(n))$ grows exponentially in n. Since $P_{\mathrm{corr}}(n)$ can be found as an affine image of a face of many other combinatorial polytopes of similar dimension, this result has been used to show that the extension complexities of polytopes such as the traveling-salesman polytope [29], certain stable set polytopes [29], certain knapsack polytopes [4, 68], and other polytopes associated with NP-hard optimization problems [4] are also not bounded polynomially.

As a first step in the proof given in [29], the authors come up with a certain submatrix of a slack matrix of $P_{\mathrm{corr}}(n)$: Given any two vectors $a, b \in \{0,1\}^n$, observe that we have

$$\begin{aligned} 1 \geq 1 - (\langle a, b\rangle - 1)^2 = 2\langle a, b\rangle - \langle a, b\rangle^2 &= 2\sum_{i=1}^{n} a_i b_i - \sum_{i,j=1}^{n} a_i a_j b_i b_j \\ &= \langle 2\,\mathrm{diag}(a) - aa^\intercal, bb^\intercal\rangle \end{aligned} \tag{4.11}$$

holds, where $\mathrm{diag}(a)$ denotes the $n \times n$-matrix with a on the diagonal and 0 everywhere else. Thus, the linear inequality

$$\langle 2\,\mathrm{diag}(a) - aa^\intercal, x\rangle \leq 1 \tag{4.12}$$

is valid for all points $x \in P_{\mathrm{corr}}(n)$. Let $\mathrm{UDISJ}(n) \in \mathbb{R}_{\geq 0}^{2^n \times 2^n}$ denote the partial slack matrix of $P_{\mathrm{corr}}(n)$ whose rows are indexed by each inequality of type (4.12) that is induced by some $a \in \{0,1\}^n$ and whose columns are indexed by all vertices $bb^\intercal$ of $P_{\mathrm{corr}}(n)$. By the computation in (4.11), choosing $x = bb^\intercal$, the left-hand side in (4.12) simply evaluates to $1 - (\langle a, b\rangle - 1)^2$. Thus, interpreting the rows and columns of $\mathrm{UDISJ}(n)$ as subsets of $\{1, \dots, n\}$, we obtain

$$\mathrm{UDISJ}(n)_{a,b} = (|a \cap b| - 1)^2.$$

for every $a, b \subseteq \{1, \dots, n\}$. Note that we have $\mathrm{UDISJ}(n)_{a,b} = 0$ if and only if $|a \cap b| = 1$.

In the second step, the authors in [29] use a bound on the rectangle-covering number of $\mathrm{UDISJ}(n)$ obtained in [21], which essentially is due to Razborov [70]. This amounts to a rather involved proof in total, leaving it unclear how "deep" the result on $P_{\mathrm{corr}}(n)$ actually is (while its great relevance is out of discussion, of course).

4.2.1 Exponential Lower Bound

Instead of using [21, 70], we give an alternative, simple combinatorial argument showing that the rectangle-covering number of $\mathrm{UDISJ}(n)$ grows exponentially in n.

Proposition 4.2.1. *For every $n \geq 0$, we have* $\text{rec-cov}(\text{UDISJ}(n)) \geq 1.5^n$.

Proof. Define $\mathcal{D}(n) := \{(a, b) \mid a, b \subseteq \{1, \dots, n\},\ a \cap b = \emptyset\}$ to be the set of disjoint pairs. We call a subset $D \subseteq \mathcal{D}(n)$ to be *valid family for n* if we have $|a \cap b'| \neq 1$ for all $(a, b), (a', b') \in D$. By induction over $n \geq 0$, we will show that every valid family for n has cardinality at most 2^n. Note that for every rectangle R of $\text{UDISJ}(n)$, the set $R \cap \mathcal{D}(n)$ is a valid family for n. Furthermore, observe that we have $|\mathcal{D}(n)| = 3^n$, as when constructing the pairs in $\mathcal{D}(n)$ one has for each $i \in \{1, \dots, n\}$ independently to choose between three possibilities (the only forbidden one being $i \in a \cap b$). Thus, since $\text{UDISJ}(n)_{a,b} > 0$ holds for all $(a, b) \in \mathcal{D}(n)$, every rectangle covering of $\text{UDISJ}(n)$ will need at least $\frac{3^n}{2^n} = 1.5^n$ rectangles, as claimed.

Clearly, since we have $\mathcal{D}(0) = \{(\emptyset, \emptyset)\}$, every valid family for n contains at most $1 = 2^0$ elements. Let now D be any valid family for $n \geq 1$. We define the following two sets:

$$D_1 := (\{(a, b) \in D \mid n \in a\} \cup \{(a, b) \in D \mid n \notin b,\ (a \cup \{n\}, b) \notin D\}),$$
$$D_2 := (\{(a, b) \in D \mid n \in b\} \cup \{(a, b) \in D \mid n \notin a,\ (a, b \cup \{n\}) \notin D\}).$$

Further, let us define the function $f\colon D \to \mathcal{D}(n-1)$ with $f(a, b) := (a \setminus \{n\}, b \setminus \{n\})$. Since $D_1 \subseteq D$ is a valid family for n and since $D_1 \subseteq \{1, \dots, n\} \times \{1, \dots, n-1\}$, $f(D_1)$ is a valid family for $n-1$. Similarly, $f(D_2)$ is also a valid family for $n-1$. Further, by the definition of D_i, f is injective on D_i for $i = 1, 2$. By the induction hypothesis, we hence obtained

$$|D_1| + |D_2| = |f(R_1)| + |f(R_2)| \leq 2^{n-1} + 2^{n-1} = 2^n.$$

Thus, it suffices to show that $D \subseteq D_1 \cup D_2$ holds. To this end, let $(a, b) \in D$. Since $a \cap b = \emptyset$, we have $(a, b) \in (\{1, \dots, n\} \times \{1, \dots, n-1\})$ or $(a, b) \in (\{1, \dots, n-1\} \times \{1, \dots, n\})$. Thus, if $n \in a \cup b$, we clearly have $(a, b) \in D_1 \cup D_2$. It remains to show that for any $(a, b) \in D$ with $n \notin a \cup b$, we cannot have $(a \cup \{n\}, b) \in D$ and $(a, b \cup \{n\}) \in D$. Indeed, this is true since, otherwise, the validity of D would imply

$$1 \neq |(a \cup \{n\}) \cap (b \cup \{n\})| = |\{n\}| = 1,$$

a contradiction. □

Together with Proposition 4.1.2 this directly implies:

Theorem 4.2.2. *For every $n \geq 1$, we have* $\mathrm{xc}(P_{\mathrm{corr}}(n)) \geq 1.5^n$.

We mention that our lower bound on $\mathrm{xc}(P_{\mathrm{corr}}(n))$ improves slightly upon the previously best known one 1.24^n following from [8]. The same holds for our bound on the rectangle-covering number of the matrix $\mathrm{UDISJ}(n)$.

4.2.2 Rectangle Coverings of UDISJ-Matrices

Using the terminology from the theory of communication complexity, the sparsity pattern of the matrix $\mathrm{UDISJ}(n)$ defines the *unique-disjointness* predicate which requires two players receiving two strings $a, b \in \{0,1\}^n$, respectively, to determine whether $\langle a, b\rangle \neq 1$, see, e.g., [43]. As the rectangle-covering number has a direct interpretation in the context of *nondeterministic communication*, it turns out that Proposition 4.2.1 implies that the *nondeterministic communication complexity* of the unique-disjointness predicate is at least $\log_2(1.5^n) \geq 0.58n$. For the background of these remarks, we refer to [53] or [43].

In this section, we establish an upper bound on the rectangle-covering number of $\mathrm{UDISJ}(n)$, which, in turn, can be interpreted as an upper bound on the nondeterministic communication complexity of the unique-disjointness predicate. However, our motivation was to investigate the limitations of $\text{rec-cov}(\mathrm{UDISJ}(n))$ as a lower bound on the extension complexity of $P_{\mathrm{corr}}(n)$.

First, note that every entry (a', b') of $\mathrm{UDISJ}(n)$ with $|a' \cap b'| \geq 2$ is contained in a rectangle of the form

$$\{a \subseteq \{1, \dots, n\} \mid i, j \in a\} \times \{b \subseteq \{1, \dots, n\} \mid i, j \in b\}$$

for some elements $i, j \in \{1, \dots, n\}$ with $i \neq j$. Thus, by defining $g(n)$ to be the smallest number of rectangles of $\mathrm{UDISJ}(n)$ needed to cover all entries (a, b) with $a \cap b = \emptyset$, which we call *disjoint pairs*, we obtain

$$\text{rec-cov}(\mathrm{UDISJ}(n)) \leq g(n) + O(n^2). \tag{4.13}$$

Next, we show that we have

$$g(n + n') \leq g(n) \cdot g(n') \tag{4.14}$$

for every $n, n' \geq 0$. To this end, let $(A_1 \times B_1), \dots, (A_k \times B_k)$ and $(A'_1 \times B'_1), \dots, (A'_\ell \times B'_\ell)$ be rectangles of $\mathrm{UDISJ}(n)$ and $\mathrm{UDISJ}(n')$ that cover all disjoint pairs, respectively. For each set $c \subseteq \{0,1\}^{n'}$ let us define $c_+ := \{i+n \mid i \in c\} \subseteq \{n+1,\dots,n+n'\}$. Consider the sets

$$R_{i,j} := \{a \cup a'_+ \mid a \in A_i,\, a' \in A'_j\} \times \{b \cup b'_+ \mid b \in B_i,\, b' \in B'_j\}$$

for $i = 1,\dots,k$ and $j = 1,\dots,\ell$. Since we have

$$|(a \cup a'_+) \cap (b \cup b'_+)| = |a \cap b| + |a'_+ \cap b'_+| = |a \cap b| + |a' \cap b'|,$$

the sets $R_{i,j}$ are rectangles of $\mathrm{UDISJ}(n+n')$. Furthermore, it is easy to check that all disjoint pairs are covered by the $R_{i,j}$'s. Since the number of these rectangles is $k \cdot \ell$ we thus established Inequality (4.14).

For $n = 2$, the set of disjoint pairs consists of the six pairs

$$\begin{aligned}
&(\emptyset,\emptyset),\ (\emptyset,\{1\}),\ (\emptyset,\{2\}),\ (\emptyset,\{1,2\}),\\
&(\{1\},\emptyset),\ (\{1\},\{2\}),\\
&(\{2\},\emptyset),\ (\{2\},\{1\}),\\
&(\{1,2\},\emptyset).
\end{aligned}$$

It can be checked that these pairs can be covered by the following three rectangles of UDISJ(2):

$$\Big\{\emptyset,\{1\}\Big\} \times \Big\{\emptyset,\{2\}\Big\}, \quad \Big\{\emptyset,\{2\}\Big\} \times \Big\{\emptyset,\{1\}\Big\}, \quad \Big\{\emptyset,\{1,2\}\Big\} \times \Big\{\emptyset,\{1,2\}\Big\}.$$

Thus, we have $g(2) \le 3$. Furthermore, it is easy to check that $g(1) \le 2$ holds. By Inequality (4.14), we now obtain

$$g(n) = g(\tfrac{n}{2} \cdot 2) \le g(2)^{\frac{n}{2}} \le \left(\sqrt{3}\right)^n$$

for even n, and

$$g(n) = g(\tfrac{n-1}{2} \cdot 2 + 1) \le g(2)^{\frac{n-1}{2}} \cdot g(1) \le \left(\sqrt{3}\right)^{n-1} \cdot 2 = \tfrac{2}{\sqrt{3}}\left(\sqrt{3}\right)^n$$

for odd n. Since $\sqrt{3} = 1.732\dots$, with Inequality (4.13) we can summarize:

Theorem 4.2.3. *There are rectangle-covering number of* UDISJ(n) *is at most* $O(1.74^n)$.

4.3 Hidden Vertices

In this section, we consider extended formulations of a special type. Investigations of this sort were already carried out in earlier work. For instance, the papers in [80] and [48] deal with extended formulations that respect symmetries of the target polytope. The work in [28] or [45] investigates the cases in which the extension polytope is restricted to the family of flow polytopes or simple polytopes, respectively.

Here, we restrict our attention to extended formulations for which every vertex of the corresponding extension polytope is projected onto a vertex of the target polytope. Many widely known extended formulations satisfy this property, see, for instance, extended formulations for the parity polytope [80, 13], the permutahedron (as a projection of the Birkhoff-polytope), the cardinality-indicating polytope [65], orbitopes [26], or spanning-tree polytopes of planar graphs [79]. Although there are not many polytopes whose extension complexity is known exactly, most of the mentioned extensions have minimum size at least up to a constant factor. Moreover, for many of these extensions there is even a one-to-one correspondence between the vertices of the extension and the vertices of the target polytope. In fact, in [61] it is shown that every 0/1-polytope of dimension at most four has a minimum-size extension of the latter type.

Clearly, a general extension might have vertices that are not projected onto vertices. Throughout this section, let us call such vertices to be *hidden vertices*. The following natural question arises: Given a polytope P, can we always find a minimum-size extension of P that has no hidden vertices? In what follows, we negatively answer the above question. Namely, in Section 4.3.1 we prove that for almost all heptagons, every minimal extension has at least one hidden vertex. In Section 4.3.2 we extend this result and construct a family of d-polytopes, such that at least $\frac{1}{9}$ of all vertices in any minimum-size extension are hidden.

4.3.1 Heptagons in General Position

In this section, we consider convex polygons with seven vertices (heptagons). Due to Shitov [76], we know that the extension complexity of any convex heptagon is at most 6. Further, it is easy to see that any affine image of a polyhedron with only 5 facets has at most 6 vertices. Thus, one obtains:

Theorem 4.3.1 (Shitov [76]). *For every convex heptagon $P \subseteq \mathbb{R}^2$ we have* $\mathrm{xc}(P) = 6$.

While Shitov's proof is purely algebraic, independently, Padrol and Pfeifle [64] established a geometric proof of this fact. In fact, they showed that any convex heptagon can be written as the projection of a 3-dimensional polytope with 6 facets. In order to get an idea of such a polytope, let us consider the following construction (which is a dual interpretation of the ideas of Padrol and Pfeifle):

Let $P \subseteq \mathbb{R}^2$ be a convex heptagon with vertices $v_1, \dots, v_7$ in cyclic order. For $i \in \{2, 3, 5, 6, 7\}$ let us set $w_i := (v_i, 0) \in \mathbb{R}^3$. Further, choose some numbers $z_1, z_4 > 0$ such that $w_1 := (v_1, z_1)$, $w_4 := (v_4, z_4)$, w_2 and w_3 are contained in one hyperplane and consider $Q' := \mathrm{conv}(\{w_1, \dots, w_7\})$. It can be shown [64] that (by possibly shifting the vertices' labels) one may assume that the convex hull of w_1, w_4 and w_6 forms a facet F of Q'. In this case, remove the defining inequality of F from an irredundant outer description of Q' and obtain a 3-dimensional polytope Q with only 6 facets whose projection is still P. For an illustration, see Figure 4.1. Note that removing the facet F results in an additional vertex that projects into the interior of P. In what follows, our argumentation does not rely on the construction described above but only on the statement of Theorem 4.3.1. Nevertheless, the previous paragraph gives an intuition why additional vertices may help in order to reduce the number of facets of an extension. We will now show that most convex heptagons force minimum-size extensions to have at least one vertex that is not projected onto a vertex. In order to avoid singular cases in which it is possible to construct minimum-size extensions without additional vertices, we only consider convex heptagons P that satisfy the following three conditions:

1. There are no four pairwise distinct vertices $u_1, \dots, u_4$ of P such that the lines $\overline{u_1 u_2}$, $\overline{u_3 u_4}$ are parallel.

2. There are no six pairwise distinct vertices $u_1, \dots, u_6$ of P, such that the lines $\overline{u_1 u_2}$, $\overline{u_3 u_4}$, $\overline{u_5 u_6}$ have a point common to all three of them.

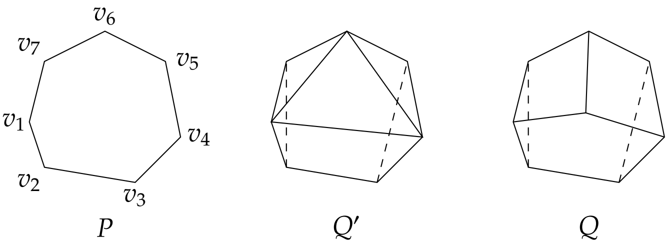

Figure 4.1: Example of the construction of a 3-dimensional extension Q with 6 facets for a heptagon P.

3. There are no seven pairwise distinct vertices $u_1, \ldots, u_7$ of P such that the intersection points $\overline{u_1 u_2} \cap \overline{u_3 u_4}$, $\overline{u_2 u_5} \cap \overline{u_4 u_6}$ and $\overline{u_3 u_7} \cap \overline{u_1 u_5}$ lie in the same line.

Here, a convex heptagon P is called to be *in general position*, if it satisfies conditions (1)–(3). In the remainder of this section, we prove the following result:

Theorem 4.3.2. *Let $P \subseteq \mathbb{R}^2$ be a convex heptagon in general position. Then any polytope that is a minimum-size extension of P has a vertex that is not projected onto a vertex of P.*

Let us fix a convex heptagon P that is in general position. In order to prove Theorem 4.3.2, let us assume, for the sake of contradiction, that there exists a polytope Q with only six facets such that Q (together with some affine map) is an extension of P and every vertex of Q is projected onto a vertex of P. Towards this end, let us first formulate two Lemmas, which we will extensively use through the whole consideration.

Lemma 4.3.3. *Let $w_1, \ldots, w_4$ be four pairwise distinct vertices of Q such that exactly one pair of them is projected onto the same vertex of P. Then, the dimension of the affine space generated by $w_1, \ldots, w_4$ equals 3.*

Proof. Let us assume the contrary and let $w_1, \ldots, w_4$ be such vertices of Q that the dimension of the corresponding affine space is at most 2. Then, the dimension of the affine space generated by the projections of $w_1, \ldots, w_4$ is at most 1 since it is the projection of the affine space

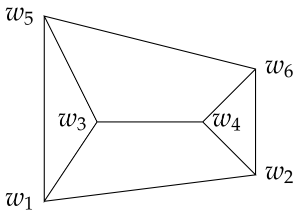

Figure 4.2: Labeling of the vertices of Γ.

generated by $w_1, \dots, w_4$, while two distinct points in this space are projected onto the same point. This implies that the projections of $w_1, \dots, w_4$, and thus three different vertices of P, lie on the same line, a contradiction. ☐

Lemma 4.3.4. *There are no six vertices $w_1, \dots, w_6$ of Q such that*

- *at most one pair of them is projected onto the same point, and*
- $\operatorname{conv}(\{w_1, \dots, w_6\})$ *is a triangular prism.*

Proof. Let $w_1, \dots, w_6$ be any six vertices of Q that form a triangular prism Γ. We claim that two of them are projected onto the same point. Otherwise, label the vertices of Γ as in Figure 4.2. The lines $\overline{w_1w_2}$ and $\overline{w_3w_4}$ are not skew, since the points w_1, w_2, w_3, w_4 lie in the same facet of Γ. By condition (1), the lines $\overline{w_1w_2}$ and $\overline{w_3w_4}$ are not parallel, since otherwise the lines $\overline{u_1u_2}$ and $\overline{u_3u_4}$ are parallel, where u_i denotes the projection of w_i for $i = 1, \dots, 6$. Thus, the lines $\overline{w_1w_2}$ and $\overline{w_3w_4}$ have a unique common point. Analogously, the lines $\overline{w_3w_4}$ and $\overline{w_5w_6}$ have a unique common point. Note, that the points $\overline{w_1w_2} \cap \overline{w_3w_4}$ and $\overline{w_3w_4} \cap \overline{w_5w_6}$ lie in the hyperplane corresponding to the facet of Γ containing w_1, w_2, w_5 and w_6, since the lines $\overline{w_1w_2}$ and $\overline{w_5w_6}$ lie in this hyperplane. Since the line $\overline{w_3w_4}$ is not contained in this hyperplane, it has at most one common point with this hyperplane, showing that $\overline{w_1w_2} \cap \overline{w_3w_4} = \overline{w_3w_4} \cap \overline{w_5w_6}$. Hence, the lines $\overline{u_1u_2}$, $\overline{u_3u_4}$ and $\overline{u_5u_6}$ have a point common to all three of them, which contradicts condition (2).

Suppose now that exactly one pair of vertices of Γ is projected onto the same point. Let us denote this point by u. Since u is a vertex

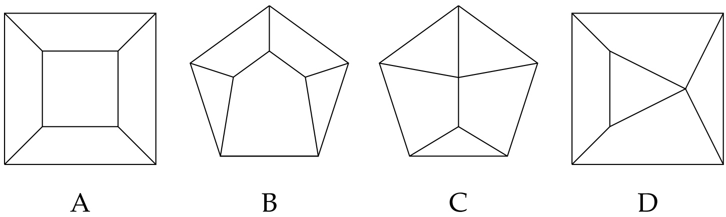

Figure 4.3: Combinatorial types of three dimensional polytopes with at most 6 facets and at least 7 vertices.

of the projection of Γ, the set of all points of Γ that project onto u forms a face of Γ. This face contains exactly two vertices of Γ, which therefore have to share an edge of Γ. But any edge of Γ is contained in a two-dimensional face of Γ with four vertices, a contradiction to Lemma 4.3.3. □

Since the polytope Q has only 6 facets, its dimension cannot exceed 5. In the case $\dim(Q) = 5$, the polytope Q would be a simplex and hence would have only 6 vertices. Note that any polytope that projects down to a convex heptagon must have at least 7 vertices. Thus, we have to consider the remaining two cases $\dim(Q) = 3$ and $\dim(Q) = 4$.

Three-Dimensional Extensions of Heptagons in General Position

In dimension 3, there are four combinatorial types of polytopes with 6 facets and at least 7 vertices (see [11]), which are illustrated in Figure 4.3.

- If Q is of type A or B, then it has 8 vertices of which exactly one pair of them is projected onto the same vertex u of P. Thus, the preimage of u induces a face of Q containing exactly two vertices of Q, hence these two vertices must share an edge. Since any edge of Q is contained in a 2-dimensional face of Q with at least 4 vertices, this yields a contradiction to Lemma 4.3.3.

- If Q is of type C, then it has 7 vertices and thus no two of them are projected on the same point. Note that six vertices of Q form a triangular prism, a contradiction to Lemma 4.3.4.

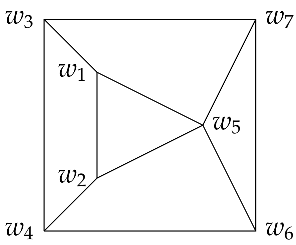

Figure 4.4: Treatment of case D.

- If Q is of type D, then due to counting, again no two of its vertices are projected on the same point. Label the vertices of Q by $w_1, \dots, w_7$ as denoted in Figure 4.4 and let u_i denote the projection of w_i for $i = 1, \dots, 7$.

  The lines $\overline{w_1 w_2}$ and $\overline{w_3 w_4}$ are not skew since they lie in the same facet of Q. By condition (1), the lines $\overline{w_1 w_2}$ and $\overline{w_3 w_4}$ are not parallel, since otherwise the lines $\overline{u_1 u_2}$ and $\overline{u_3 u_4}$ are parallel. Thus, we have that the lines $\overline{w_1 w_2}$ and $\overline{w_3 w_4}$ have a unique common point. Analogously, one obtains that $\overline{w_1 w_5} \cap \overline{w_3 w_7}$ and $\overline{w_2 w_5} \cap \overline{w_4 w_6}$ each consists of a single point.

  Moreover, the points $\overline{w_1 w_2} \cap \overline{w_3 w_4}$, $\overline{w_1 w_5} \cap \overline{w_3 w_7}$ and $\overline{w_2 w_5} \cap \overline{w_4 w_6}$ belong to the intersection of the hyperplane generated by w_1, w_2, w_5 and the hyperplane generated by w_3, w_4, w_6, w_7; i.e. the points $\overline{w_1 w_2} \cap \overline{w_3 w_4}$, $\overline{w_1 w_5} \cap \overline{w_3 w_7}$ and $\overline{w_2 w_5} \cap \overline{w_4 w_6}$ lie in the same line. Hence, $u_1, \dots, u_7$ violate condition (3).

Four-Dimensional Extensions of Heptagons in General Position

In dimension 4, every polytope with exactly 6 vertices is the convex hull of a union of a 4-simplex with one point. Dualizing this observation yields that every 4-polytope with exactly 6 facets is combinatorially equivalent to a 4-simplex Δ intersected with one closed affine half-space H. The combinatorial structure of such an intersection is completely defined by the number k of the vertices of Δ lying on the boundary of H and the number t of the vertices of Δ lying outside of H.

In this case, the total number of vertices of Q equals

$$(5 - t) + (5 - k - t)t,$$

i.e. $5 - t$ vertices of Δ are vertices of Q and all other vertices of Q are intersections of the edges between the t vertices of Δ lying outside of H and $5 - k - t$ vertices of Δ lying strictly inside of H. Since the number of vertices of Q should be at least 7, there are five possibilities for the pair (k, t), namely $(0, 1)$, $(1, 1)$, $(0, 2)$, $(1, 2)$ and $(0, 3)$. In order to finally rule out the existence of Q, by Lemma 4.3.4, it suffices to show that in each of these cases, Q contains a triangular prism as a facet of which at most one pair of vertices is projected onto the same point.

- $(k, t) \in \{(0, 1), (1, 1), (1, 2), (0, 3)\}$: In these cases, the number of vertices of Q is at most 8 and thus, at most one pair of vertices of Q is projected onto the same point. There exists a facet F of Δ such that none of its vertices lies on the boundary of H and one or two of its vertices lie outside of H. Consider the facet $F' := F \cap H$ of Q, which is a 3-simplex intersected with one half-space. In particular, by the choice of F, F' is a triangular prism.

- $(k, t) = (0, 2)$: In this case, Q has exactly 9 vertices. Let w be a vertex of Q such that its projection coincides with the projection of another vertex of Q. Let F be a facet of Δ that does not contain w. Then at most one pair of vertices of the facet $F \cap H$ of Q is projected onto the same point. To finish the proof, note that for $(k, t) = (0, 2)$ the intersection of every facet of Δ with the half-space H is a triangular prism.

4.3.2 Higher-Dimensional Constructions

We now show that heptagons in general position are not the only polytopes that force minimum-size extensions to have hidden vertices. In fact, Theorem 4.3.6 yields families of polytopes P in arbitrary dimensions such that for any polytope Q that is a minimum-size extension of P, at least a constant fraction of the vertices of Q are hidden. To this end, we first investigate the structure of extended formulations for polytopes of the form $P \times [0, 1]$.

Lemma 4.3.5. *Let $P \subseteq \mathbb{R}^p$, $Q \subseteq \mathbb{R}^q$ be polytopes such that Q together with some affine map $\pi\colon \mathbb{R}^q \to \mathbb{R}^{p+1}$ is a minimum-size extension of $P \times [0,1]$. Then, the sets*

$$F_0 := \left\{y \in Q \mid \pi(y)_{p+1} = 0\right\}, \quad F_1 := \left\{y \in Q \mid \pi(y)_{p+1} = 1\right\}$$

together with π are both minimum-size extensions of P.

Proof. Note that F_0 and F_1 are both extensions of P and proper faces of Q. Let k denote the number of facets of Q and t be the extension complexity of P. Clearly, we have that $k \le t+2$ holds. For $i \in \{0,1\}$ let $f_i \ge t$ be the number of facets of F_i and let us define the sets

$$\begin{aligned} \mathcal{C}_i &:= \{F \mid F \text{ is a facet of } Q,\ F_i \subseteq F\}\,, \\ \mathcal{D}_i &:= \{F \mid F \text{ is a facet of } Q,\ F_i \cap F = \emptyset\}\,. \end{aligned}$$

It is straightforward to see that

$$f_i \le k - |\mathcal{C}_i| - |\mathcal{D}_i| \le (t+2) - |\mathcal{C}_i| - |\mathcal{D}_i| \tag{4.15}$$

holds. Thus, it remains to show that $|\mathcal{C}_i| + |\mathcal{D}_i| \ge 2$ holds for $i = 0,1$. Clearly, this inequality holds if neither F_0 is a facet nor is F_1, since this implies $|\mathcal{C}_i| \ge 2$ for $i = 0,1$.

Towards this end, by symmetry, it remains to consider the case that F_0 is a facet of Q. Since $|\mathcal{C}_i| \ge 1$ for $i = 0,1$, it is enough to show that in this case we have $|\mathcal{D}_i| \ge 1$ for $i = 0,1$. Indeed, since F_0 and F_1 are disjoint, we obtain $F_0 \in \mathcal{D}_1$ and thus $|\mathcal{D}_1| \ge 1$. Due to $t \le f_1$ and Inequality (4.15), it holds that $|\mathcal{C}_1| + |\mathcal{D}_1| \le 2$ and hence $|\mathcal{C}_1| = 1$. Thus, F_1 has to be a facet, too. Moreover, the facet F_1 is in $\mathcal{D}_0$, and thus $|\mathcal{D}_0| \ge 1$. □

Theorem 4.3.6. *Let P be a convex heptagon in general position and Q a polytope that is a minimum-size extension of $P \times [0,1]^d$. Then, for at least $\frac{1}{9}$ of the vertices of Q, we have that none of them is projected onto a vertex of $P \times [0,1]^d$.*

Proof. We will prove the statement by induction over $d \ge 0$. In the case of $d = 0$, by Theorem 4.3.2 and its proof, we know that Q has at most 9 vertices and that at least one of them is not projected onto a vertex of P.

For $d \geq 1$, let us define $P' := P \times [0,1]^{d-1} \subseteq \mathbb{R}^{d+1}$. Let $Q \in \mathbb{R}^q$ be a polytope such that Q together with some affine map $\pi \colon \mathbb{R}^q \to \mathbb{R}^{d+2}$ is a minimum-size extension of $P \times [0,1]^d = P' \times [0,1] \subseteq \mathbb{R}^{d+2}$. Observe that the vertex set of $P' \times [0,1]$ is the cartesian product of the vertex set of P' and the set $\{0,1\}$. Let us partition the set V of vertices of Q into the three sets

$$\begin{aligned} V_0 &:= \{y \in V \mid \pi(y)_{d+2} = 0\}, \\ V_1 &:= \{y \in V \mid \pi(y)_{d+2} = 1\}, \\ V_\star &:= \{y \in V \mid 0 < \pi(y)_{d+2} < 1\}. \end{aligned}$$

Clearly, none of the vertices in $V_\star$ is projected onto a vertex of $P' \times [0,1]$. By Lemma 4.3.5, $\operatorname{conv}(V_0)$ and $\operatorname{conv}(V_1)$ are minimum-size extensions of P'. By induction, for at least $\frac{1}{9}(|V_0| + |V_1|)$ vertices in $V_0 \cup V_1$, we have that none of them is projected onto a vertex of $P' \times [0,1]$. Thus, the number of vertices that are not projected onto a vertex is at least $\frac{1}{9}(|V_0| + |V_1|) + |V_\star| \geq \frac{1}{9}|V|$. □

5

Describing Polytopes: Open Questions

In this section, we state a few open questions related to topics covered in this chapter. Some of these questions have been already posed at scientific meetings but were only rarely published in written form.

Extension Complexity of the Spanning-Tree Polytope

In the beginning of this chapter we have reviewed Martin's [56] extended formulation for the spanning-tree polytope $P_{\text{sp.trees}}(G)$ of a graph $G = (V, E)$, which has size $\mathcal{O}(|V| \cdot |E|)$.

Problem 1. Is the bound $\text{xc}(P_{\text{sp.trees}}(G)) \leq \mathcal{O}(|V| \cdot |E|)$ asymptotically tight for general graphs?

Problem 2. Let K_n be the complete graph on n nodes. Do we have $\text{xc}(P_{\text{sp.trees}}(K_n)) = \Theta(n^3)$?

We remark that this thesis provides at least some insights concerning these problems. For instance, we showed that the extension complexities of the spanning-tree polytope and the nonempty subgraph polytope $P^{\star}_{\text{sub}}(G)$ coincide up to an additive error of $\mathcal{O}(|E|)$ (see Theorem 3.2.2). Furthermore, $\text{xc}(P_{\text{sp.trees}}(G))$ is an upper bound (again up to an error of $\mathcal{O}(|E|)$) on the extension complexities of independence polytopes of all count matroids $\mathcal{M}_{m,\ell}(G)$ with $m \geq \ell$ (see the discussion below Theorem 3.3.3) as well as on the cut dominant $P_{\text{cut+}}(G)$ (see Theorem 3.4.1). Thus, any non-trivial lower bound on the extension complexity of one of these polytopes would already yield progress towards answering the above questions.

Finally, note that Proposition 4.1.4 provides limitations on several general combinatorial bounds on the extension complexity when applied to the spanning-tree polytope.

Extensions of Cartesian Products

Let P, Q be a pair of nonempty polytopes. Clearly, we have that

$$\operatorname{xc}(P \times Q) \leq \operatorname{xc}(P) + \operatorname{xc}(Q) \tag{5.1}$$

holds. However, the author is not aware of any such pair satisfying this inequality strictly. As an example, the proof of Lemma 4.3.5 shows that $\operatorname{xc}(P \times [0,1]) = \operatorname{xc}(P) + 2$ holds for every nonempty P. Thus, since we have $\operatorname{xc}([0,1]^d) = 2d$, see [30], this implies that Inequality (5.1) holds as an equation whenever $Q = [0,1]^d$ for some d. Using the same argumentation, this even holds for all combinatorial cubes Q. Furthermore, it can be shown that Inequality 5.1 holds as an equation if Q is a simplex [66].

Problem 3. Does $\operatorname{xc}(P \times Q) = \operatorname{xc}(P) + \operatorname{xc}(Q)$ hold for every pair P, Q of nonempty polytopes?

Rectangle Coverings and Nonnegative Rank of UDISJ(n)

In Section 4.2 we showed that the rectangle-covering number (and hence the nonnegative rank) of UDISJ(n) is at least 1.5^n. However, we also argued that rec-cov(UDISJ(n)) can be bounded by 1.74^n times some constant. We remark that this upper bound can be improved whenever we have $\sqrt[k]{g(k)} < \sqrt{3}$ for some integer k and g being defined as in Section 4.2.2. (Computations indicate that this seems to be not the case for $k = 3, 4$.)

Problem 4. What is the asymptotical growth of rec-cov(UDISJ(n))?

As $\mathrm{UDISJ}(n)$ has 2^n columns (and rows), its nonnegative rank is at most 2^n. It can be checked by hand that this estimation is tight for $n = 1, 2, 3$. Moreover, the author is not aware of any n for which the nonnegative rank of $\mathrm{UDISJ}(n)$ is less than 2^n.

Problem 5. Is the nonnegative rank of $\mathrm{UDISJ}(n)$ equal to 2^n for every n?

We refer to [34] for computational results supporting an affirmative answer to the above question, see [34, Conj. 4].

Recall that the nonnegative rank of $\mathrm{UDISJ}(n)$ is a lower bound on the extension complexity of the correlation polytope $P_{\mathrm{corr}}(n)$. To our knowledge, the best possible upper bound on $\mathrm{xc}(P_{\mathrm{corr}}(n))$ is 2^n due to the fact that $P_{\mathrm{corr}}(n)$ has 2^n vertices. We believe that this bound is best-possible.

Problem 6. Does $\mathrm{xc}(P_{\mathrm{corr}}(n)) = 2^n$ hold for every n?

Extension Complexity Exponential in The Dimension

We have seen that the extension complexity of $P_{\mathrm{corr}}(n)$ grows exponentially in n. As the dimension of $P_{\mathrm{corr}}(n)$ is $\frac{1}{2}(n^2 - n)$, this means that $\mathrm{xc}(P_{\mathrm{corr}}(n))$ grows only exponentially in the square root of the dimension of $P_{\mathrm{corr}}(n)$. By the work of Rothvoß [72], the same applies for the traveling-salesman polytope $P_{\mathrm{TSP}}(n)$ and the matching polytope $P_{\mathrm{match}}(n)$. On the other hand, in [71] it is shown that there *exists* a family of 0/1-polytopes whose extension complexities grow exponentially in their dimensions. However, we are not aware of any explicit family with this property.

Problem 7. Find an explicit family of 0/1-polytopes $P_1, P_2, \ldots$ of strictly growing dimensions such that

$$\mathrm{xc}(P_i) \geq 2^{c \cdot \dim(P_i)}$$

holds for all i, where $c \in \mathbb{R}$ is some absolute constant.

Independence Polytopes of Matroids

In Section 3.3 we gave polynomial-size extended formulations for independence polytopes for a few classes of matroids. It is a natural question to ask for which other classes of matroids there are (similar) constructions of polynomial-size extended formulations for the associated independence polytopes.

One main ingredient we used to establish polynomial-size extended formulations for independence polytopes of regular matroids, was the fact that whenever a matroid $\mathcal{M}$ can be decomposed by means of 1-, 2- and 3-sums, the extension complexity of $P(\mathcal{M})$ can be bounded by the sum of the extension complexities of the leaf nodes' independence polytopes. However, not many classes of matroids are known that admit decompositions using only 1-, 2- and 3-sums and starting from simple building blocks – as in case of regular matroids. As an obvious generalization of regular matroids, linear matroids over $\mathbb{F}_2$ do not seem to have such decompositions.

Problem 8. For any fixed field $\mathbb{F}$, does $\mathrm{xc}(P(\mathcal{M}))$ grow polynomially (in the dimension) for every $\mathbb{F}$-linear matroid $\mathcal{M}$?

As already mentioned, we know from [71] that there *exists* a family of independence polytopes of matroids whose extension complexities grow exponentially in their dimension. However, no such family is known explicitly. We would like to highlight a class of matroids that might contain interesting candidates: Given an undirected graph G, let $\mathcal{M}_{\mathrm{match}}(G)$ be the *matching matroid* [54] of G whose independent sets are the subsets of nodes of G that can be covered by some matching. It is easy to see that the extension complexity of $\mathcal{M}_{\mathrm{match}}(G)$ can be bounded in terms of the extension complexity of the matching polytope of G. Clearly, this neither implies a polynomial-size upper bound (for general graphs) nor any lower bound on $\mathcal{M}_{\mathrm{match}}(G)$.

Problem 9. Does there exist a family of infinitely many graphs G such that $\mathrm{xc}(\mathcal{M}_{\mathrm{match}}(G))$ grows superpolynomially in their dimensions?

Clearly, given a candidate matroid $\mathcal{M}$, the question arises, how to prove a non-trivial lower bound on $\mathrm{xc}(P(\mathcal{M}))$. We remark that Proposition 4.1.3 shows that lower bounds that are dominated by the rectangle-covering bound cannot provide superpolynomial bounds in this case.

Part II

Describing Integer Points in Polytopes

6

Describing Integer Points in Polytopes: Definitions & Background

In the previous chapter we have seen that there are sets X associated to prominent combinatorial-optimization problems, for which the polytope $\mathrm{conv}(X)$ cannot be described by (projections of) polyhedra having polynomially many facets in the dimension of X. One such example is the vertex-set of the correlation polytope, which consists of all matrices $bb^\intercal$ with $b \in \{0,1\}^n$. However, these points are also the integer points of the polytope

$$\Big\{x \in [0,1]^n \mid x_{i,j} \leq x_{i,i} \quad \text{for all } i \in \{1,\dots,n\}, \\ x_{i,i} + x_{j,j} \leq x_{i,j} + 1 \text{for all } i,j \in \{1,\dots,n\},\ i \neq j\Big\}, \tag{6.1}$$

which has only $O(n^2)$ many facets. Another prominent polytope having exponential extension complexity is the traveling-salesman polytope of the complete graph $K_n = (V_n, E_n)$ on n nodes. Let us denote the set of its vertices, i.e., the set of characteristic vectors of (edge sets of) hamiltonian cycles in K_n, by $X_{\mathrm{STSP}}(n) \subseteq \{0,1\}^{E_n}$. It is well-known that $X_{\mathrm{STSP}}(n)$ is the set of integer points of the *subtour polytope*

$$P_{\mathrm{subtour}}(n) := \Big\{x \in [0,1]^{E_n} \mid x(E(U)) \leq |U| - 1 \text{ for all } \emptyset \neq U \subset V_n, \\ x(\delta(v)) = 2 \text{ for all } v \in V_n\Big\}, \tag{6.2}$$

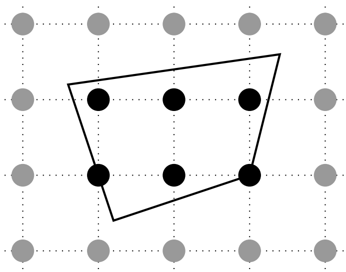

Figure 6.1: The polygon is a relaxation for the black integer points.

where $E(U)$ denotes the set of edges that have both endnodes in U, and $\delta(v)$ denotes the set of edges incident to v. In contrast to (6.1), the subtour polytope has exponentially many facets in n. One may wonder whether any polyhedron whose set of integer points coincides with $X_{\mathrm{STSP}}(n)$ necessarily needs to have exponentially many facets, while for many other sets associated to (hard) combinatorial-optimization problems like, e.g., the maximum-clique, the maximum-satisfiability or facility-location problem, polynomial size descriptions as in (6.1) are readily at hand.

This chapter addresses questions concerning properties and in particular sizes of polyhedra whose set of integer points coincides with some given set X. For this purpose, given a set $X \in \mathbb{Z}^d$, we call a polyhedron $P \subseteq \mathbb{R}^d$ satisfying $P \cap \mathbb{Z}^d = X$ a *relaxation of* X. Our main interest focuses on the *relaxation complexity* of a set X, which we define as the smallest number of facets of any relaxation of X and denote by $\mathrm{rc}(X)$. In what follows, we will mainly restrict ourselves to sets X such that $\mathrm{conv}(X)$ is a polyhedron and $X = \mathrm{conv}(X) \cap \mathbb{Z}^d$ holds. We will call such sets *polyhedral*.

The work reported about in this chapter had its origins in the question about the behavior of $\mathrm{rc}(X_{\mathrm{STSP}}(n))$, which, to our initial slight surprise, apparently has not been treated before. Moreover, except for a paper by Jeroslow [41], the author is not aware of any reference that deals with a similar quantity as the relaxation complexity. In his paper, for a set $X \subseteq \{0,1\}^d$ of binary vectors, Jeroslow introduces the term *index* of X (short: $\operatorname{ind}(X)$), which is defined as the smallest number of inequalities needed to separate X from the remaining points in $\{0,1\}^d$. Thus, the notion of relaxation complexity can be seen as a natural extension of the index with respect to general subsets of $\mathbb{Z}^d$. Clearly, we have that $\operatorname{ind}(X) \leq \operatorname{rc}(X)$ holds for all sets $X \in \{0,1\}^d$. On the other hand, as we will briefly discuss in Section 7.2.1, both quantities differ at most by an additive term of $d+1$. As the main result in his paper, Jeroslow shows that 2^{d-1} is an upper bound on $\operatorname{ind}(X)$, which is attained by the set of binary vectors of length d that contain an even number of ones, see Sections 7.4.2 and 8.3.3. We generalize his idea of bounding the index of a set $X \subseteq \{0,1\}^d$ from below to provide lower bounds on the relaxation complexity of general X, see Section 8.2.2.

This allows us to provide exponential lower bounds on the relaxation complexities of sets associated to problems that are variants of the traveling-salesman, the spanning-tree, or the T-join problem. In particular, we show that the asymptotic growth of $\operatorname{rc}(X_{\mathrm{STSP}}(n))$ is $2^{\Theta(n)}$. In this sense, the exponentially large subtour elimination formulation thus is asymptotically smallest possible.

Before we investigate lower bounds on sizes of relaxations, we discuss several questions concerning the construction of (small) relaxations.

Outline In Section 7.1, we put our concept of relaxations into context with more general integer-programming formulations that involve auxiliary variables. Furthermore, instead of the explicit size of a relaxation P, we consider the existence of a polynomial-time algorithm for solving linear programs over P as a measure of its complexity. It turns out that both models are already powerful enough to allow for descriptions of small complexity for most sets X associated to reasonable combinatorial-optimization problems. Thus, this part shows why we focus on our particular notion of relaxations.

In Section 7.2, we give examples for constructions of relaxations having different properties, which serves as a simple starting point to get familiar with questions asked (and being answered) in this chapter. There, among others, we will provide an answer to a conjecture recently posed by Padberg [63], see Section 7.2.4.

Section 7.3 briefly addresses questions concerning sizes of coefficients in linear descriptions of (minimum size) relaxations, which are not captured by our notion of relaxation complexity. General upper bounds on the relaxation complexities of sets $X \subseteq \mathbb{Z}^d$ in terms of $|X|$ and d are established in Section 7.4. For this purpose, we give improved bounds on recent Helly-type results by Aliev et al. [1]. In Section 7.5, we discuss algorithmic aspects of computing the relaxation complexity of a given set.

The second part of this chapter is concerned with new proof techniques providing lower bounds on sizes of relaxations. To begin with, we work out lower bounds on the relaxation complexities of two basic sets, namely the vertex sets of $[0,1]^d$ and the standard simplex, respectively. In Section 8.2, we consider bounds on sizes of general relaxations. To this end, we first prove that the relaxation complexity of a random subset of $\{0,1\}^d$ grows exponentially in d. In order to find explicit sets X having that property, we then introduce the concept of *hiding sets*, which turns out to be a powerful technique to provide lower bounds on $\mathrm{rc}(X)$. Finally, in Section 8.3, we use this approach to give exponential lower bounds on the sizes of relaxations for concrete structures that occur in many practical integer programming formulations.

Parts of this chapter form the base of a joint publication with Volker Kaibel:

- Volker Kaibel and Stefan Weltge. Lower Bounds on the Sizes of Integer Programs without Additional Variables. *Math. Program. Ser. B*, 154(1-2):407–425, 2015.

Some of the results – in particular those of Section 7.4 – emerged from valuable discussions with Gennadiy Averkov.

7

Describing Integer Points in Polytopes: Constructions

7.1 Extended Relaxations

While our notion of relaxations only captures integer-programming models in the ambient space of X, many prominent formulations of combinatorial-optimization problems are of the form

$$\max / \min \left\{ \langle c, x \rangle \mid x \in \mathbb{Z}^d,\ \exists y \in \mathbb{Z}^q : Ax + By \le b \right\},$$

where the vector y consists of additional variables and the set X of feasible points is described via $X = \{x \in \mathbb{Z}^E \mid \exists y \in \mathbb{Z}^q : \ Ax + By \le b\}$. In the case of $X = X_{\mathrm{STSP}}(n)$, there even exist classical formulations of that type for which the system $Ax + By \le b$ consists of polynomially many linear inequalities, see, e.g. [58] or [33]. It turns out that in fact every reasonable combinatorial-optimization problem admits polynomial-size descriptions of this type:

Following Schrijver's proof [73, Thm. 18.1] of the fact that integer programming is $\mathcal{NP}$-hard, one finds that for any language $\mathcal{L} \subseteq \{0, 1\}^*$ that is in $\mathcal{NP}$, there is a polynomial p such that for any $k > 0$ there is a system $Ax + By \le b$ of at most $p(k)$ linear inequalities and $m \le p(k)$

auxiliary variables with

$$\left\{x \in \{0,1\}^k \mid x \in \mathcal{L}\right\} = \left\{x \in \{0,1\}^k \mid \exists y \in \{0,1\}^m : Ax + By \leq b\right\}.$$

However, for (families of) sets X that we usually encounter in combinatorial optimization, we even have polynomial-time algorithms for deciding whether a given vector is contained X. Further, suppose we are given a *boolean circuit* and let $X \subseteq \{0,1\}^d$ be the set of inputs that evaluate to true. It is straightforward to model the outputs of all intermediate gates in terms of additional variables and linear inequalities: For inputs $y_1, y_2 \in \{0,1\}$, the resulting output y_3 of a, say, OR-gate is the unique solution $y_3 \in [0,1]$ of the system $y_1 \leq y_3$, $y_2 \leq y_3$, $y_3 \leq y_1 + y_2$. A crucial property of these constraints is that if the inputs have 0/1-values, then y_3 is also implicitly forced to take its value in $\{0,1\}$. It is straightforward to make analogous observations for AND-gates and NOT-gates, see, e.g., [80, p. 445]. Since for every language $\mathcal{L} \in \mathcal{P}$ there exists a polynomial-time algorithm to construct polynomial-size boolean circuits that decide $\mathcal{L}$, see, e.g., [3, pp. 109–110], we conclude:

Proposition 7.1.1. *Let $X_d \subseteq \{0,1\}^d$ be a family of sets such that the membership problem "Given $x \in \{0,1\}^d$, is x in X_d?" is in $\mathcal{NP}$. Then there exists a polynomial p such that for any d there is a system $Ax + By \leq b$ of at most $p(d)$ linear inequalities and $m \leq p(d)$ auxiliary variables with*

$$X_d = \left\{x \in \{0,1\}^d \mid \exists y \in \mathbb{Z}^m : Ax + By \leq b\right\}.$$

If the membership problem is even in $\mathcal{P}$, then there exist such systems without integrality constraints on the auxiliary variables y.

Coming back to the exponential size of the subtour-elimination relaxation for $X_{\mathrm{STSP}}(n)$, it might be argued that the number of inequalities is not the right measure of complexity since it is still possible to optimize linear functions over the subtour-elimination relaxation in polynomial-time. Interestingly, the mere existence of a relaxation over which optimization can be performed in polynomial time is, again, nothing very special:

Proposition 7.1.2. *Let $X_d \subseteq \{0,1\}^d$ be a family of sets such that the membership problem "Given $x \in \{0,1\}^d$, is x in X_d?" is in $\mathcal{P}$. Then there exists a family of relaxations R_d for X_d such that linear programming over R_d can be done in polynomial time.*

Proof. By Proposition 7.1.1, we know that for each d there exists a system $Ax + By \leq b$ of polynomially many linear inequalities such that $X_d = \{x \in \{0,1\}^d \mid \exists y \in \mathbb{R}^m : Ax + By \leq b\}$. As mentioned in the above argumentation, such systems even can be constructed by a polynomial-time algorithm. Thus, setting

$$R'_d := \{(x, y) \mid Ax + By \leq b,\ x \in [0,1]^d,\ y \in \mathbb{R}^m\},$$
$$R_d := \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^m : (x, y) \in R'_d\},$$

gives us the desired relaxations R_d. Indeed, given $c \in \mathbb{Q}^d$ we have that

$$\max\{\langle c, x\rangle \mid x \in R_d\} = \max\{\langle c, x\rangle \mid \exists y : (x, y) \in R'_d\},$$

where the latter problem can be solved in time polynomially bounded in d and the encoding length of c. □

Thus, the models used in Proposition 7.1.1 and Proposition 7.1.2 are already powerful enough to allow for descriptions of small complexity for sets X associated to reasonable combinatorial-optimization problems. As we will see later, this is not always the case for our notion of relaxations, which motivates our interest in understanding this kind of descriptions.

7.2 Examples

For most sets X associated to combinatorial-optimization problems, it is a simple task come up with *some* relaxation for X. However, this may become a non-intuitive task when asking for relaxations of smallest possible size or satisfying given properties. This section is meant to serve as a starting point and illustrates such issues by means of a few examples.

7.2.1 Hypercube

When we say that X is associated to some combinatorial-optimization problem, it usually consists of characteristic vectors of feasible objects and hence is a subset of $\{0,1\}^d$ for some d. Most relaxations of such sets are designed by using the inequalities $0 \leq x_i \leq 1$ for $i = 1, \ldots, d$ in order to separate all non-binary points from X, while all remaining inequalities serve the purpose of cutting off those binary points that do not belong to X. In practice, the more challenging task is to find a proper system of linear inequalities fulfilling the latter task. Furthermore, as $[0,1]^d$ describes the convex hull of $\{0,1\}^d$ and consists of only $2d$ facets, it seems to be the natural relaxation of $\{0,1\}^d$ for these purposes. However, we show that the set of binary points can be described by using only $d+1$ linear inequalities.

Lemma 7.2.1. *For $d \geq 1$, we have*

$$\{0,1\}^d = \Big\{x \in \mathbb{Z}^d \mid x_k \leq 1 + \sum_{i=k+1}^{d} 2^{-i} x_i \text{ for all } k = 1,\dots,d,\ x_1 + \sum_{i=2}^{d} 2^{-i} x_i \geq 0\Big\}.$$

Proof. It is clear that $\{0,1\}^d$ is contained in the set of the right-hand side. Let $x \in \mathbb{Z}^d$ be any integer point satisfying

$$x_k \leq 1 + \sum_{i=k+1}^{d} 2^{-i} x_i \tag{7.1}$$

for all $k = 1,\dots,d$ as well as

$$x_1 + \sum_{i=2}^{d} 2^{-i} x_i \geq 0. \tag{7.2}$$

First, we claim that $x_i \leq 1$ holds for all $i = 1,\dots,d$: Otherwise, suppose that $k \in \{1,\dots,d\}$ is the largest index for which we have $x_k > 1$. Then we obtain

$$x_k \overset{(7.1)}{\leq} 1 + \sum_{i=k+1}^{d} 2^{-i} x_i \leq 1 + \sum_{i=k+1}^{d} 2^{-i} < 2,$$

a contradiction. Furthermore, we see that x_1 is nonnegative since we have (due to $x_i \leq 1$ for all i)

$$x_1 \overset{(7.2)}{\geq} -\sum_{i=2}^{d} 2^{-i} x_i \geq -\sum_{i=2}^{d} 2^{-i} > -1.$$

It remains to show that $x_i \geq 0$ holds for all $i = 2, \ldots, d$. To this end, suppose that we have $x_j \leq -1$ for some $j \in \{2, \ldots, d\}$ and $x_i \geq 0$ for all $i < j$. Then we claim that $x_i = 0$ holds for all $i < j$: Otherwise, let k be the largest $k < j$ such that $x_k > 0$ holds (and hence $x_k = 1$). By Inequality (7.1), we would obtain

$$
\begin{aligned}
1 = x_k \leq 1 + \sum_{i=k+1}^{d} 2^{-i} x_i &= 1 + 2^{-j} x_j + \sum_{i=j+1}^{d} 2^{-i} x_i \\
&\leq 1 + 2^{-j} \cdot (-1) + \sum_{i=j+1}^{d} 2^{-i} < 1.
\end{aligned}
$$

Thus, we have $x_i \geq 0$ for all $i < j$, and hence, by Inequality (7.2), we deduce

$$
0 \leq x_1 + \sum_{i=2}^{d} 2^{-i} x_i = 2^{-j} x_j + \sum_{i=j+1}^{d} 2^{-i} x_i \leq 2^{-j} \cdot (-1) + \sum_{i=j+1}^{d} 2^{-i} < 0,
$$

a contradiction. □

Thus, the relaxation complexity of $\{0,1\}^d$ is at most $d+1$. In Section 8.1.1, we will see this construction is best possible.

7.2.2 Cross Polytope

Let $X_{\text{cross}}(d)$ be the set that consists of the origin of $\mathbb{Z}^d$ and the standard unit vectors in $\mathbb{Z}^d$ together with their negatives. Its convex hull is known as the *cross polytope* and has exactly 2^d facets, which are defined by inequalities of the form

$$
\sum_{i \in S} x_i - \sum_{j \notin S} x_j \leq 1,
$$

where S is some subset of $\{1, \ldots, d\}$. It is easy to come up with a relaxation of $X_{\text{cross}}(d)$ that consists of only $O(d^2)$ many facets, as given by the polytope

$$
\begin{aligned}
\Big\{ x \in [-1,1]^d \mid & -1 \leq x_i + x_j \leq 1 \text{ for all } i \neq j, \\
& -1 \leq x_i - x_j \leq 1 \text{ for all } i \neq j \Big\}.
\end{aligned}
$$

However, we show that the relaxation complexity of $X_{\text{cross}}(d)$ can even be linearly bounded in d:

Proposition 7.2.2. *For $d \geq 4$, we have*

$$X_{\text{cross}}(d) = \Big\{ x \in \mathbb{Z}^d \mid -1 \leq x_j - \sum_{i \neq j} x_i \leq 1 \textit{ for all } j = 1, \dots, d \Big\}.$$

In particular, $\operatorname{rc}(X_{\text{cross}}(d)) \leq 2d$ *holds for all* $d \geq 4$.

Proof. Let $Y \subseteq \mathbb{Z}^d$ denote the set on the right-hand side. Clearly, we have that $X_{\text{cross}}(d)$ is contained in Y. Let y be some point in Y. Since we have $Y = -Y$, we may assume that $\sum_{i=1}^d y_i \geq 0$ holds. For every $j \in \{1, \dots, d\}$, we have

$$y_j = \frac{1}{2}(y_j + y_j) \overset{y \in Y}{\geq} \frac{1}{2}(y_j + \sum_{i \neq j} y_i - 1) = \frac{1}{2}(\sum_{i=1}^d y_i - 1) \geq -\frac{1}{2},$$

and hence $y_j \geq 0$ (since $y_j \in \mathbb{Z}$). Since further Q is invariant under coordinate permutation, we may assume that $y_1 \geq \cdots \geq y_d$ holds. We obtain

$$\sum_{i=1}^{d-2} y_i \leq \sum_{i=1}^{d-2} y_i + y_{d-1} - y_d = \sum_{i \neq d} y_i - y_d \overset{y \in Y}{\leq} 1,$$

which implies $y_1 \leq 1$ and $y_2 = \cdots = y_d = 0$. Thus, y is either the origin or the first standard unit vector, which are both contained in $X_{\text{cross}}(d)$. □

Note that all inequalities defining the relaxation in Proposition 7.2.2 are facet-defining for $X_{\text{cross}}(d)$. We do not know whether this construction is best possible.

7.2.3 Restricting to Facet-Defining Inequalities

Many known relaxations for sets associated with combinatorial-optimization problems are defined by linear inequalities of which, preferably, most of them are facet-defining for $\operatorname{conv}(X)$. Clearly, this has important practical reasons since such formulations are tightest possible in some sense. However, if one is interested in a relaxation that has as few number of facets as possible, one cannot only use facet-defining inequalities of $\operatorname{conv}(X)$: In the previous section, we have seen that $\operatorname{rc}(\{0,1\}^d) = d + 1$ holds, whereas by removing any of the cube's inequalities the remaining (unbounded) polyhedron contains integer points that are not contained in $\{0,1\}^d$. Nevertheless, the restriction to facet-defining inequalities seems to be not too hard:

Proposition 7.2.3. *Let $X \subseteq \mathbb{Z}^d$ be polyhedral and $\operatorname{rc}_F(X)$ the smallest number of facets of any relaxation of X whose facet-defining inequalities are also facet-defining for $\operatorname{conv}(X)$. Then, $\operatorname{rc}_F(X) \leq \dim(X) \cdot \operatorname{rc}(X)$.*

Proof. By Carathéodory's Theorem, any facet-defining inequality of a relaxation R for X can be replaced (and possibly strengthened) by $\dim(X)$ many facet-defining inequalities of $\operatorname{conv}(X)$. The resulting polyhedron is still a relaxation for X. □

7.2.4 Subtours versus Combs

One of our introductory examples has been the set $X_{\text{STSP}}(n)$, which consists of the set of characteristic vectors of hamiltonian cycles in the (undirected) complete graph $K_n = (V_n, E_n)$ on n nodes. The subtour polytope (see (6.2)) is a well-known relaxation of this set, which, in the field of combinatorial optimization, serves as a prototype of a relaxation having interesting properties. Recall that the subtour polytope consists of all points $x \in [0,1]^{E_n}$ that satisfy the *degree constraints*

$$x(\delta(v)) = 2 \quad \text{for all } v \in V, \tag{7.3}$$

as well as the *subtour-elimination constraints*

$$x(E(U)) \leq |U| - 1 \quad \text{for all } \emptyset \neq U \subset V_n. \tag{7.4}$$

On the one side, the number of constraints in (7.4) grows exponentially in n. However, this cannot be avoided when constructing a relaxation for $X_{\text{STSP}}(n)$ as we will see later. On the other side, the subtour polytope has less facets than $\operatorname{conv}(X_{\text{STSP}}(n))$, i.e., the traveling-salesman polytope. The inequalities $0 \leq x_e \leq 1$ for all $e \in E_n$ together with all inequalities in (7.4) can be shown to be facet-defining for $\operatorname{conv}(X_{\text{STSP}}(n))$. Furthermore, it is well-known that they can be separated efficiently yielding a polynomial-time algorithm for optimizing linear functions over the subtour polytope. We refer to Grötschel & Padberg [37, 38] for background information on these facts.

Following the literature, it seems that all relaxations for $X_{\mathrm{STSP}}(n)$ that have been investigated, are the subtour polytope and refinements of it, i.e., intersections of the subtour polytope with polyhedra defined by further facet-defining inequalities of the traveling-salesman polytope. Thus, as a reader one gets the impression that every relaxation for $X_{\mathrm{STSP}}(n)$ that is only defined by facet-defining inequalities of $\mathrm{conv}(X_{\mathrm{STSP}}(n))$ has to be based on the subtour-elimination constraints. For instance, in [63] Padberg claims that every system of facet-defining inequalities of $\mathrm{conv}(X_{\mathrm{STSP}}(n))$ defining a relaxation of $X_{\mathrm{STSP}}(n)$ contains the subtour-elimination constraints (modulo linear combinations of the degree constraints). He points out that this "needs a formal proof" and leaves "such a formal proof (or disproof?) as 'food for thought' for the younger talents in our field" (see [63, p. 47]). In what follows, we disprove the conjecture.

For this purpose, we make use of the notion of a *comb*, which is defined as a set of vertices $H \subseteq V_n$ (*handle*) together with vertex sets $T_1, \dots, T_p \subseteq V_n$ (*teeth*) such that $p \geq 3$ is odd, all teeth are disjoint, and the sets $H \cap T_j$ and $T_j \setminus H$ are nonempty for $j = 1, \dots, p$. Now the associated *comb inequality* reads

$$x(E(H)) + \sum_{j=1}^{p} x(E(T_j)) \leq |H| + \sum_{j=1}^{p} |T_j| - \tfrac{3p+1}{2}, \tag{7.5}$$

and can be shown to be valid for $X_{\mathrm{STSP}}(n)$, see [37, Prop. 1.3]. In particular, every such inequality even defines a facet of the traveling-salesman polytope, see [38, Thm. 6.2]. Furthermore, it can be checked that these facets are distinct from those that are defined by subtour-elimination constraints. We are ready to give an alternative relaxation for $X_{\mathrm{STSP}}(n)$:

Proposition 7.2.4. *For $n \geq 7$, let $P \subseteq \mathbb{R}^{E_n}$ be the polytope defined by $0 \leq x_e \leq 1$ for all $e \in E$, all degree constraints (7.3), and the comb inequalities (7.5) for combs with exactly three teeth, i.e., $p = 3$. Then we have $P \cap \mathbb{Z}^d = X_{\mathrm{STSP}}(n)$.*

Proof. Clearly, we have $X_{\mathrm{STSP}}(n) \subseteq P$. For the reverse inclusion, observe that each $x \in P \cap \mathbb{Z}^d$ is the characteristic vector of some set of edges $F \subseteq$

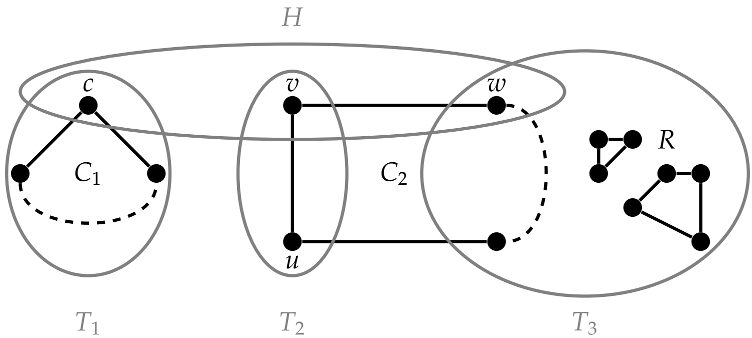

Figure 7.1: Illustration of the comb constructed in the Proof of Proposition 7.2.4.

E_n for which every node of V_n has degree two. Thus F is a collection of node-disjoint cycles and hence it suffices to show that F consists of exactly one cycle.

For the sake of contradiction, assume that F contains two distinct cycles C_1, C_2 and denote by $R := F \setminus (C_1 \cap C_2)$ the set of the remaining edges in F. Since $n \geq 7$, we may assume that $|C_2| + |R| \geq 4$ holds, otherwise change the labels of C_1 and C_2. Let $c \in V(C_1)$ be an arbitrary node of C_1 and $\{u, v\}, \{v, w\} \subseteq C_2$ be two distinct edges in C_2 for some nodes $u, v, w \in V(C_2)$.

Let us define a comb with three teeth in the following way: $H := \{c, v, w\}$, $T_1 := V(C_1)$, $T_2 := \{u, v\}$ and $T_3 := (V(C_2) \cup V(R)) \setminus \{u, v\}$. Note that $|T_3 \setminus H| = |C_2| - 2 + |R| \geq 1$. As an illustration, see Figure 7.2.4. Now the left-hand side of (7.5) evaluates to

$$
\begin{aligned}
x(E(H)) + \sum_{j=1}^{3} x(E(T_j)) \\
&= \underbrace{|E(H) \cap F|}_{=1} + \underbrace{|E(T_1) \cap F|}_{=|C_1|} + \underbrace{|E(T_2) \cap F|}_{=1} + \underbrace{|E(T_3) \cap F|}_{=|C_2|-2+|R|} \\
&= |C_1| + |C_2| + |R|,
\end{aligned}
$$

which is greater than the right-hand side of (7.5):

$$\begin{aligned}|H| + |T_1| + |T_2| + |T_3| - \tfrac{3p+1}{2} &= 3 + |C_1| + 2 + (|C_2| - 2 + |R|) - 5 \\ &= |C_1| + |C_2| + |R| - 2.\end{aligned}$$

Thus, the corresponding comb inequality is violated by x, a contradiction to $x \in P$. □

We remark that Carr [12] showed that the comb inequalities for fixed p can be separated in polynomial time. Thus, it is possible to optimize linear functions over the relaxation of Proposition 7.2.4 in polynomial time.

7.3 Coefficients

In the previous section, we already discussed the (application-oriented) restriction to relaxations that are defined by facet-defining (for $\text{conv}(X)$) inequalities only. Another technical requirement concerns the coefficients in outer descriptions of relaxations: In prominent formulations, most linear inequalities serve the purpose of enforcing logical or simple combinatorial constraints. For that reason, usually the number different values that are taken by coefficients is very small. In particular, many formulations consist of linear inequalities whose coefficients even take only values in $\{-1, 0, 1\}$. In practical modeling, for the sake of numerical stability and easier handling in many components of branch-and-bound algorithms, formulations that make use of only few and simple numbers are often preferred over descriptions whose complexity is hidden in the coefficients.

7.3.1 Encoding Lengths

The relaxation complexity does not involve sizes of coefficients in outer descriptions of (minimum-size) relaxations. In fact, we are not aware of any bounds on the coefficients' sizes of minimum-size relaxations for general polyhedral sets. However, we show that in order to separate sets of 0/1-points from each other, there is no need to use linear inequalities with coefficients of arbitrary complexity. As a measure of complexity, we use the notion of *encoding length* of a rational number $r \in \mathbb{Q}$, which (is often simply called the "size" of r and) captures the number of bits needed to encode r in its binary representation. For a precise definition and discussion, see, e.g., Schrijver [73, Sec. 3.2].

Proposition 7.3.1. *There exists a constant $c \in \mathbb{R}$ such that for any real vector $a \in \mathbb{R}^d$ and any real number $\gamma \in \mathbb{R}$ there is a rational vector $a' \in \mathbb{Q}^d$ and a rational number $\gamma' \in \mathbb{Q}^d$ satisfying*

- $\{x \in \{0,1\}^d \mid \langle a, x\rangle \le \gamma\} = \{x \in \{0,1\}^d \mid \langle a', x\rangle \le \gamma'\}$, *and*
- *the encoding lengths of γ' and each entry in a' can be bounded by $c \cdot d^2$.*

Proof. We may assume that $X := \{x \in \{0,1\}^d \mid \langle a, x\rangle \le \gamma\}$ is not empty. By setting $\bar{x} := \arg\max\{\langle a, x\rangle \mid x \in X\}$, let us define the affine isomorphism $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ via $\varphi(x) := x - \bar{x}$. Clearly, we now have that

$$\langle a, x\rangle \le \gamma \iff \langle a, \varphi(x)\rangle \le 0$$

holds for all $x \in \{0,1\}^d$. Thus, the polyhedron

$$\begin{aligned} P := \Big\{\tilde{a} \in \mathbb{R}^d \mid \langle y, \tilde{a}\rangle \le 0 \text{ for all } y \in \varphi(X), \\ \langle y, \tilde{a}\rangle \ge 1 \text{ for all } y \in \varphi(\{0,1\}^d \setminus X)\Big\} \end{aligned}$$

is not empty. Since P can be described by a system of linear inequalities whose coefficients are in $\{-1, 0, 1\}$, we know that there exists a rational point $a' \in P$ whose entries have encoding length that is bounded by $c' \cdot d^2$ for some absolute constant c', see [73, Chap. 10]. For every point $x \in \{0,1\}^d$, by the definition of a', we now have that

$$\begin{aligned} \langle a, x\rangle \le \gamma \iff \langle a, \varphi(x)\rangle \le 0 &\iff \langle a', \varphi(x)\rangle \le 0 \\ &\iff \langle a', x - \bar{x}\rangle \le 0 \iff \langle a', x\rangle \le \langle a', \bar{x}\rangle \end{aligned}$$

holds. Set $\gamma' := \langle a', \bar{x}\rangle$ and observe that the encoding length of γ' is at most the sum of the encoding lengths of the entries in a' times some absolute constant. □

The above statement tells us that if we are given a set $X \subseteq \{0,1\}^d$, then we can always find a rational relaxation $\tilde{R}$ for X whose number of facets is close to $\mathrm{rc}(X)$ and the encoding lengths of coefficients in a suitable outer description of $\tilde{R}$ can be quadratically bounded in d. Indeed, let $\{x \in \mathbb{R}^d \mid Ax \leq b, Cx = d\}$ be a minimum-size relaxation of X. Perturb the inequalities in $Ax \leq b$ according to Proposition 7.3.1, remove the system $Cx = d$ and call the obtained polyhedron R'. If we now choose $S \subseteq \mathbb{R}^d$ to be any relaxation for $\{0,1\}^d$ and H to be the affine hull of X, we obtain that $\tilde{R} := R' \cap S \cap H$ is a relaxation for X. By Lemma 7.2.1, the number of inequalities we thus have to add, i.e., the number of facets of S, can be assumed to be at most $d+1$. Note that the encoding lengths of the coefficients in the outer description of 7.2.1 grow only linearly in d. Since X only consists of 0/1-vectors, its affine hull H can be described by a system of linear equations whose coefficients' encoding lengths can also be bounded quadratically in the dimension, see [73, Chap. 10].

Corollary 7.3.2. *There exists a constant $c \in \mathbb{R}$ such that for every set $X \subseteq \{0,1\}^d$ there is a relaxation R such that R of X that*

- *has at most* $\mathrm{rc}(X) + d + 1$ *facets, and*
- *can be described by a system of linear inequalities and equations whose coefficients have encoding lengths bounded by $c \cdot d^2$.*

7.3.2 Role of Rationality

Focusing on minimum-size relaxations, one may ask the question whether it may help to use irrational coordinates in the description of a relaxation. In the case of X being finite, it is easy to see that one does not lose too much when restricting to rational relaxations only:

Proposition 7.3.3. *Let $X \subseteq \mathbb{Z}^d$ be finite and $\mathrm{rc}_{\mathbb{Q}}(X)$ be the smallest number of facets of any rational relaxation for X. Then,* $\mathrm{rc}_{\mathbb{Q}}(X) \leq \mathrm{rc}(X) + \dim(X) + 1$.

Proof. Since X is finite, there exists a rational simplex $\Delta \subseteq \mathbb{R}^d$ of dimension $\dim(X)$ such that X is contained in Δ. Let R be any relaxation of X having f facets and set $B := (\mathbb{Z}^d \setminus X) \cap \Delta$. Since $B \cap R = \emptyset$ and B consists of only finitely many points, we are able to slightly perturb the facet-defining inequalities of R in order to obtain a polyhedron $\tilde{R}$ such that $B \cap \tilde{R} = \emptyset$ and $\tilde{R}$ is rational. Now $\tilde{R} \cap \Delta$ is still a relaxation for

X, which is rational and has at most $f + (\dim(\Delta) + 1) = f + \dim(X) + 1$ facets. □

However, we are not aware of any polyhedral set X where $\mathrm{rc}(X) < \mathrm{rc}_{\mathbb{Q}}(X)$ holds. Let $\Delta_d := \{\mathbb{O}, \mathbb{e}_1, \ldots, \mathbb{e}_d\}$ be the set of vertices of the standard simplex. While (the convex hull of) Δ_d can be easily described by the $d+1$ linear inequalities $x_i \geq 0$ for $i = 1, \ldots, d$ and $x_1 + \cdots + x_d \leq 1$, we even do not know whether $\mathrm{rc}(\Delta_d) < d + 1$ holds. Note that any relaxation R for Δ_d that has less than $d+1$ facets has to be unbounded. Hence, if R was rational, it would contain a rational ray and hence infinitely many integer points, which shows $\mathrm{rc}_{\mathbb{Q}}(\Delta_d) = d + 1$.

When proving a lower bound on the relaxation complexity of $\{0,1\}^d$ in Section 8.1.1, we will use the fact that for any line $L(c) := \{\lambda c \in \mathbb{R}^d \mid \lambda \in \mathbb{R}\}$ with $c \in \mathbb{R}^d \setminus \{\mathbb{O}\}$, the set $[0,1]^d + L(c)$ contains infinitely many integer points. Unfortunately, such a statement is not true for the general simplex:

Consider the 5-dimensional simplex

$$S := \mathrm{conv}\,\{\mathbb{O}, \mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3, \mathbb{e}_1 + \mathbb{e}_3 + \mathbb{e}_4, \mathbb{e}_2 + \mathbb{e}_3 + \mathbb{e}_5\} \subseteq \mathbb{R}^5.$$

Setting $c := (0, 0, 0, 1, \sqrt{2})$, we claim that the polyhedron $S + L(c)$ does not contain any other integer points than those in S. To this end, let $p + \lambda \cdot c$ be integral for some $p \in S$ and some $\lambda \in \mathbb{R}$. Since the first three entries of c are zero, the first three entries of p have to be integer. It is easy to check that this forces p to be one of the vertices of S and hence to be an integer vector. Since then both λ and $\lambda\sqrt{2}$ are integers, this implies $\lambda = 0$.

Since S does not contain other integer points than its vertices, we can apply a unimodular transformation and obtain a direction $c' \in \mathbb{R}^5$ such that $R = \mathrm{conv}(\Delta_5) + L(c')$ is indeed an unbounded relaxation for Δ_5. However, it can be verified that R has more than 6 facets in this case. In Section 8.1.2, we follow another argumentation to obtain at least some lower bound on the relaxation complexity of Δ_d.

7.4 General Upper Bounds

In this section, we provide upper bounds on the relaxation complexity of general polyhedral sets $X \in \mathbb{Z}^d$. As the sets we usually encounter have large cardinality (especially when compared to the dimension), these bounds will be stated in terms of $|X|$ and d.

7.4.1 General Case

Let us first calculate a simple upper bound on the relaxation complexity of X by considering a very natural relaxation of X – its convex hull. In order to estimate the number of facets of $\operatorname{conv}(X)$, we can make use of McMullen's Upper Bound Theorem [57], which, as a special case, states that the number of facets of a d-dimensional polytope with n vertices can be bounded by

$$\binom{n-\lceil d/2\rceil}{\lfloor d/2\rfloor}+\binom{n-1-\lceil (d-1)/2\rceil}{\lfloor (d-1)/2\rfloor}\le 2\cdot n^{d/2}.$$

As $\operatorname{conv}(X)$ has at most $|X|$ vertices, we obtain that

$$\operatorname{rc}(X)\le 2\cdot |X|^{d/2} \tag{7.6}$$

holds for every polyhedral set $X\subseteq\mathbb{Z}^d$.

On the Number of Vertices

Estimating the number of vertices of $\operatorname{conv}(X)$ by $|X|$ might be naive. In fact, we show that the number of vertices of $\operatorname{conv}(X)$ grows only sublinear in $|X|$ if d is fixed. For this purpose, we make use of a classical statement due to Kannan & Lovász [50], which gives a bound on the "lattice width" of a set of integer points:

Lemma 7.4.1 (see [50, Thm. 4.1])**.** *There is a constant $c\in\mathbb{R}$ such that for every polyhedral set $X\subseteq\mathbb{Z}^d$ there exists a vector $v\in\mathbb{Z}^d\setminus\{\mathbb{O}\}$ with*

$$\max_{x\in X}\langle v,x\rangle-\min_{x\in X}\langle v,x\rangle\le c\cdot d^2\cdot(|X|+1)^{\frac{1}{d}}.$$

Proposition 7.4.2. *There is a constant $c\in\mathbb{R}$ such that for every polyhedral set $X\subseteq\mathbb{R}^d$ the number of vertices of* $\operatorname{conv}(X)$ *is at most $c^d\cdot(|X|+1)^{1-1/d}$.*

Proof. Let c_0 be the constant of Lemma 7.4.1 and observe that there exists another constant $c\ge 2$ such that

$$2(c_0d^2)^{\frac{1}{d-1}}\le c \tag{7.7}$$

holds for every $d \geq 2$. For integers n, d with $n \geq 0$ and $d \geq 1$ let $f(n,d)$ be the maximum number of vertices of $\operatorname{conv}(X)$ of any polyhedral set $X \subseteq \mathbb{Z}^d$ with $|X| = n$. We have to show that $f(n,d) \leq c^d \cdot (n+1)^{(d-1)/d}$ holds for every n, d. We proceed by induction over $d \geq 1$. Observe that the claim is true for $d = 1$ since every polytope in $\mathbb{R}$ has at most two vertices and $c \geq 2$. Let now $d \geq 2$ and $X \subseteq \mathbb{Z}^d$ any finite polyhedral set. Choose v as in Lemma 7.4.1 and define the set

$$J := \{j \in \mathbb{Z} \mid \langle v, x\rangle = j \text{ for some } x \in X\}.$$

By the choice of v, we have

$$|J| \leq c_0 \cdot d^2 \cdot (|X|+1)^{1/d} + 1 \leq 2c_0 \cdot d^2 \cdot (|X|+1)^{1/d}, \tag{7.8}$$

where the second inequality relies on the fact that we may assume $c_0 \geq 1$. Furthermore, we clearly have

$$|J| \leq |X|. \tag{7.9}$$

Now for every $j \in J$ define $X_j := \{x \in X \mid \langle v, x\rangle = j\}$. Each such X_j is contained in a rational hyperplane H and hence $H \cap \mathbb{Z}^d$ can be bijectively mapped to $\mathbb{Z}^{d-1}$ by some coordinate projection. Since X_j is polyhedral, we have that the number of vertices of $\operatorname{conv}(X_j)$ is at most $f(|X_j|, d-1)$. Furthermore, note that we have $X = \cup_{j\in J} X_j$ and that every vertex x of $\operatorname{conv}(X)$ with $x \in X_j$ is also a vertex of $\operatorname{conv}(X_j)$. Thus, setting $n := |X|$ and $n_j := |X_j|$ for all $j \in J$, we obtain that the number of vertices of $\operatorname{conv}(X)$ is at most

$$\begin{aligned}
\sum_{j\in J} f(n_j, d-1) \leq c^{d-1} \cdot \sum_{j \in J} (n_j+1)^{\frac{d-2}{d-1}} &\leq c^{d-1} \cdot |J| \cdot \left(\tfrac{n}{|J|} + 1\right)^{\frac{d-2}{d-1}} \\
&\leq c^{d-1} \cdot |J| \cdot \left(2\tfrac{n}{|J|}\right)^{\frac{d-2}{d-1}} \\
&= 2^{\frac{d-2}{d-1}} \cdot c^{d-1} \cdot |J|^{\frac{1}{d-1}} \cdot n^{\frac{d-2}{d-1}},
\end{aligned}$$

where the first inequality follows by induction, the second inequality follows from the concavity of $x \mapsto (x+1)^{\frac{d-2}{d-1}}$ and the fact $n = \sum_{j \in J} n_j$, and the third inequality via (7.9). Thus, estimating $|J|$ with Inequality (7.8), we obtain that the number of vertices of $\operatorname{conv}(X)$ is at most

$$\begin{aligned} 2^{\frac{d-2}{d-1}} \cdot c^{d-1} \cdot \left(2c_0 \cdot (n+1)^{1/d} \cdot d^2\right)^{\frac{1}{d-1}} \cdot n^{\frac{d-2}{d-1}} & \\ &= 2\left(c_0 \cdot d^2\right)^{\frac{1}{d-1}} \cdot c^{d-1} \cdot (n+1)^{\frac{1}{(d-1)d}} \cdot n^{\frac{d-2}{d-1}} \\ &\leq c^d \cdot (n+1)^{\frac{1}{(d-1)d}+\frac{d-2}{d-1}} \\ &= c^d \cdot (n+1)^{\frac{d-1}{d}}, \end{aligned}$$

where the inequality follows from the definition of c, see (7.7). □

We remark that the bound of Proposition 7.4.2 is tight for fixed d, in the sense that Bárány & Larman [6] showed that if X is the set of integer points in a large ball, then the number of vertices of $\operatorname{conv}(X)$ behaves as $|X|^{1-2/d}$ times some constant depending on d.

Clearly, our estimation on the number of vertices of $\operatorname{conv}(X)$ only slightly improves the bound on the relaxation complexity of X presented in the first part of this section. In particular, this approach still yields a bound that, for fixed $d \geq 5$, grows superlinear in $|X|$. In what follows, we present a different strategy to give an upper bound on $\operatorname{rc}(X)$ that, for fixed d, grows only sublinear in $|X|$.

Helly-type Bounds

Let us consider a quantity that is related to the relaxation complexity. For every pair of nonnegative integers n, d with $d \geq 1$, let $g(n, d)$ denote the smallest number such that for any relaxation $R = \{x \in \mathbb{R}^d \mid Ax \leq b\}$ for some set $X \subseteq \mathbb{Z}^d$ with $|X| = n$ there exists a subset S of the rows of A, b with the property that $\{x \in \mathbb{R}^d \mid A_{S,*}x \leq b_{S,*}\}$ is also a relaxation of X. Clearly, $g(n, d)$ is an upper bound on the relaxation complexity of every polyhedral set $X \subseteq \mathbb{Z}^d$ with $|X| = n$. Recently, the function g was investigated by Aliev et al. [1] who showed that $g(n, d)$ can be bounded from above by $\lceil 2(n+1)/3 \rceil 2^d - 2\lceil 2(n+1)/3 \rceil + 2$. Note that this yields

$$\operatorname{rc}(X) \leq (|X| + 1) \cdot 2^d$$

for every polyhedral set $X \subseteq \mathbb{Z}^d$ and hence already improves on our previous bounds. Here, using a very similar argumentation as given in [1], we derive an alternative bound on $g(n, d)$ that performs slightly better if n is much larger than d:

Theorem 7.4.3. *There is a constant $c \in \mathbb{R}$ such that*

$$g(n, d) \leq c^d \cdot n^{1-\frac{1}{d}}$$

holds for all n, d with $n \geq c^{d^2}$.

In particular, we obtain that

$$\mathrm{rc}(X) \leq c^d \cdot |X|^{1-\frac{1}{d}}$$

holds for every sufficiently large polyhedral set $X \subseteq \mathbb{Z}^d$.

Proof of Theorem 7.4.3. Let $a_1, \dots, a_m \in \mathbb{R}^d$ and $b_1, \dots, b_m \in \mathbb{R}$ such that the system of linear inequalities

$$\langle a_1, x\rangle \leq b_1, \dots, \langle a_m, x\rangle \leq b_m \tag{7.10}$$

has exactly n integer solutions, which we denote by $X \subseteq \mathbb{Z}^d$. We may further assume that there exist points $x_1, \dots, x_m \in \mathbb{Z}^d$ such that every x_i violates $\langle a_i, x\rangle \leq b_i$ but satisfies all other inequalities in (7.10). Let us define

$$Y := \mathrm{conv}(X \cup \{x_1, \dots, x_m\}) \cap \mathbb{Z}^d.$$

Since Y is finite, we can increase every b_i and obtain a new right-hand side b'_i such that

$$X = \{y \in Y \mid \langle a_1, y\rangle < b'_1, \dots, \langle a_m, y\rangle < b'_m\}$$

holds. We can further iteratively increase every b'_i and obtain a right-hand side b''_i such that

$$X = \{y \in Y \mid \langle a_1, y\rangle < b''_1, \dots, \langle a_m, y\rangle < b''_m\}$$

holds and there exist $v_1, \dots, v_m \in Y$ such that every v_i satisfies $\langle a_i, v_i\rangle = b''_i$ and $\langle a_j, v_i\rangle < b_i$ for all $j \neq i$. This is possible due to the choice of $x_1, \dots, x_m$ and the definition of Y. (Note that v_i does not necessarily have to coincide with x_i.) Now consider $P := \mathrm{conv}(\{v_1, \dots, v_m\})$. It is easy to verify that all v_i's are pairwise distinct and that each v_i is a vertex of P. Furthermore, by the construction of $v_1, \dots, v_m$, we have

$$P \cap \mathbb{Z}^d \setminus \{v_1, \dots, v_m\} \subseteq X$$

and hence $|P \cap \mathbb{Z}^d| \leq |X| + m \leq n + m$. By Proposition 7.4.2, we obtain

$$m \leq c_0^d \cdot (n + m + 1)^{1-\frac{1}{d}} \tag{7.11}$$

for some absolute constant $c_0 > 0$.

Defining $c := 2c_0$, according to the assumption on the size of n we may assume that $n \geq c^{d^2}$ holds. We claim that this implies $m \leq n + 1$. Otherwise, Inequality (7.11) yields $m \leq c_0^d \cdot (2m)^{1-1/d}$, which is equivalent to $m \leq c_0^{d^2} \cdot 2^{d-1}$ and hence implies

$$n + 1 \leq m \leq c_0^{d^2} \cdot 2^{d-1} \leq (2c_0)^{d^2} = c^{d^2},$$

a contradiction. Thus, we have $m \leq n + 1 \leq 2n$. Together with Inequality (7.11) we finally obtain

$$m \leq c^d \cdot (2(n+1))^{1-\frac{1}{d}} \leq c^d \cdot n^{1-\frac{1}{d}}.$$

□

7.4.2 Binary Points

Many sets X we are interested in are not arbitrary (polyhedral) subsets of $\mathbb{Z}^d$ but subsets of $\{0,1\}^d$. As mentioned in the introduction to this chapter, Jeroslow [41] showed that for any set $X \subseteq \{0,1\}^d$, one needs at most 2^{d-1} many linear inequalities in order to separate X from $\{0,1\}^d \setminus X$. If $P \subseteq \mathbb{R}^d$ is a polyhedron such that $P \cap \{0,1\}^d = X$ holds, then, in order to construct a relaxation for X, we only need to additionally separate all points $\mathbb{Z}^d \setminus \{0,1\}^d$ from X. Clearly, this can be done by intersecting P with a relaxation for $\{0,1\}^d$, which, by Lemma 7.2.1, can be assumed to have only $d + 1$ facets. Thus, we derived that

$$\mathrm{rc}(X) \leq 2^{d-1} + d + 1 \tag{7.12}$$

holds for every set $X \subseteq \{0,1\}^d$. As we have $|X| \le 2^d$ for every such set, the estimation in (7.12) already improves on the general bounds of the previous section.

In what follows, however, we will refine Jeroslow's idea to obtain a more specific bound on the relaxation complexity that depends on the cardinality of X. As he argues, every point $y \in \{0,1\}^d$ can be separated from X by the "canonical cut"

$$\sum_{\substack{i=1\\ y_i=0}}^{d} x_i + \sum_{\substack{j=1\\ y_j=1}}^{d} (1 - x_j) \ge 1,$$

which is a valid inequality for all points $x \in X$ but is violated for $x = y$. This shows that we need at most $2^d - |X|$ linear inequalities in order to separate all points $\mathbb{Z}^d \setminus \{0,1\}^d$ from X. While this bound performs well if X is large, it is weak if X is small:

Lemma 7.4.4. *For every nonempty set $X \subseteq \{0,1\}^d$ there exists a system of at most $|X|$ linear inequalities that are valid for X, such that every point $y \in \{0,1\}^d \setminus X$ violates at least one of these inequalities.*

Proof. We proceed by induction over $d \ge 1$ and observe that the claim is true for $d = 1$. For $d \ge 2$ we consider the sets $X_0 := \{x \in \{0,1\}^{d-1} \mid (x,0) \in X\}$ and $X_1 := \{x \in \{0,1\}^{d-1} \mid (x,1) \in X\}$. Note that we have $X = (X_0 \times \{0\}) \cup (X_1 \times \{1\})$.

Suppose first that one of the two sets is empty. In this case, we may assume that X_1 is empty but X_0 is not. By the induction hypothesis, there exist vectors $a'_1, \ldots, a'_k \in \mathbb{R}^{d-1}$ and numbers $\gamma_1, \ldots, \gamma_k$ with $k \le |X_0|$ such that

$$X_0 = \{x \in \{0,1\}^{d-1} \mid \langle a'_i, x\rangle \le \gamma_i \text{ for } i = 1, \ldots, k\}$$

holds. Let us choose $M > 0$ such that $\langle a'_i, x\rangle + M > \gamma_i$ holds for all $i = 1, \ldots, k$ and all $x \in X_1$. Furthermore, let $a_i \in \mathbb{R}^d$ arise from a'_i by appending M to the last coordinate. By construction, we have that X can be described via

$$X = \{x \in \{0,1\}^d \mid \langle a_i, x\rangle \le \gamma \text{ for } i = 1, \ldots, k\},$$

where $k \le |X_0| = |X|$, as claimed.

We are left with the case that both X_0 and X_1 are nonempty. Again by the induction hypothesis, there exist $a'_1, \dots, a'_k \in \mathbb{R}^{d-1}$, $\gamma_1, \dots, \gamma_k \in \mathbb{R}$ with $k \le |X_0|$, and $c'_1, \dots, c'_\ell \in \mathbb{R}^{d-1}$, $\delta_1, \dots, \delta_\ell \in \mathbb{R}$ with $\ell \le |X_1|$ such that

$$\begin{aligned} X_0 &= \{x \in \{0,1\}^{d-1} \mid \langle a'_i, x\rangle \le \gamma_i \text{ for } i = 1, \dots, k\}, \\ X_1 &= \{x \in \{0,1\}^{d-1} \mid \langle c'_i, x\rangle \ge \delta_i \text{ for } i = 1, \dots, \ell\} \end{aligned}$$

holds. Let us choose $M > 0$ such that $\langle a'_i, x\rangle \le \gamma_i + M$ holds for all $i = 1, \dots, k$ and all $x \in X_1$ and such that $\langle c'_i, x\rangle \ge \delta_i - M$ holds for all $i = 1, \dots, \ell$ and all $x \in X_0$. Again, let a_i arise from a'_i by appending $-M$ to the last coordinate, and let c_i arise from c'_i in the same way. By construction, we obtain that

$$\begin{aligned} X = \{x \in \{0,1\}^d \mid &\langle a_i, x\rangle \le \gamma_i \text{ for } i = 1, \dots, k, \\ &\langle c_i, x\rangle \ge \delta_i \text{ for } i = 1, \dots, \ell\} \end{aligned}$$

holds. The claim follows since we have $k + \ell \le |X_0| + |X_1| = |X|$. □

Together with the previous paragraph and the fact that the relaxation complexity of $\{0,1\}^d$ is at most $d + 1$ (see Lemma 7.2.1) we summarize:

Theorem 7.4.5. *For every set $X \subseteq \{0,1\}^d$ we have*

$$\operatorname{rc}(X) \le \min\{|X|, 2^d - |X|\} + d + 1.$$

7.5 Computational Complexity

Given a finite polyhedral set $X \subseteq \mathbb{Z}^d$, one may ask the question whether its relaxation complexity is computable. Unfortunately, we are not aware of any finite algorithm computing the exact value of $\operatorname{rc}(X)$ in general dimension. Nevertheless, there is a simple algorithm computing the relaxation complexity up to an additive term of $\dim(X) + 1$.

Proposition 7.5.1. *There exists an algorithm that, for given finite polyhedral X, computes a relaxation of X whose number of facets is at most* $\operatorname{rc}(X) + \dim(X) + 1$.

Proof. We describe the algorithm on a high-level:

1. Compute a $\dim(X)$-dimensional simplex $\Delta \subseteq \mathbb{R}^d$, which contains the set X.

2. Enumerate all points in $Y := \Delta \cap \mathbb{Z}^d \setminus X$.

3. Let $\mathcal{C}$ be the set of all subsets $C \subseteq Y$ for which there exists a *separating inequality* $\langle a^C, x\rangle \leq b^C$ that is valid for X but violated for every point in C.

4. Compute the smallest number of sets $C_1, \ldots, C_k$ in $\mathcal{C}$ needed to cover Y.

5. Return the polyhedron $\Delta \cap \{x \in \mathbb{R}^d \mid \langle a^{C_i}, x\rangle \leq b^{C_i} \text{ for } i = 1, \ldots, k\}$.

Note that the sets of $\mathcal{C}$ in Step 3 can be computed by checking feasibility for auxiliary systems of linear inequalities.

By construction, the polyhedron Q returned in Step 5 is a relaxation of X. Let P be any relaxation of X and denote its facet-defining inequalities by $\langle a^i, x\rangle \leq b^i$ for $i = 1, \ldots, \ell$. Then for each i, the set $C'_i := \{y \in Y \mid \langle a^i, y\rangle > b^i\}$ is contained in $\mathcal{C}$. Furthermore, since P is a relaxation, the sets $C'_1, \ldots, C'_\ell$ cover Y. Thus, defining k as in Step 4, we obtain that P has at least $\ell \geq k$ facets. On the other side, the relaxation Q has at most $k + \dim(X) + 1$ facets. □

Clearly, any implementation of the steps in the proof of Proposition 7.5.1 is expected to result in a very time-consuming algorithm. However, it is very doubtful whether there exist efficient algorithms for answering the above questions, since many related (potentially simpler) problems turn out to be computationally difficult. For instance, consider the problem

(Q1) "Given a matrix $A \in \mathbb{Q}^{m\times d}$, a vector $b \in \mathbb{Q}^m$, and a polyhedral (finite) set $X \subseteq \mathbb{Z}^d$, is $\{x \in \mathbb{R}^d \mid Ax \leq b\}$ a relaxation of X?".

This decision problem can be easily shown to be hard: In the particular case of X being the empty set, the above question, in negated form, reduces to the question

(Q2) "Given a matrix $A \in \mathbb{Q}^{m\times d}$ and a vector $b \in \mathbb{Q}^m$, does $\{x \in \mathbb{R}^d \mid Ax \leq b\}$ contain an integer point?", which is well-known to be an $\mathcal{NP}$-hard problem [32, Sec. A6]. Hence, the problem of answering (Q1) is co$\mathcal{NP}$-hard.

Many algorithmic questions concerning integer points in polyhedra get tractable when restricting to a fixed dimension. This is also the case for problem (Q1): It is a simple exercise to check that, by basically solving $\mathcal{O}(\binom{|X|}{\dim(X)})$ systems of linear equations, one can compute a system of linear inequalities $Dx \leq d$ in time polynomial in the encoding length of X such that $\operatorname{conv}(X) = \{x \in \mathbb{R}^d \mid Dx \leq d\}$ holds. Now $P := \{x \in \mathbb{R}^d \mid Ax \leq b\}$ is a relaxation of X if and only if (i) X is contained in P, and (ii) if one has

$$\max\{D_{i,*}x \mid x \in P \cap \mathbb{Z}^d\} \leq d_i.$$

for every row $D_{i,*}x \leq d_i$ of $Dx \leq d$. While checking (i) is a trivial task, (ii) can be verified by solving an integer program over P for each row of $Dx \leq d$. Since integer programming in fixed dimension can be done in polynomial time [42], we obtain:

Proposition 7.5.2. *For fixed d, problem (Q1) can be solved in time polynomial in the encoding lengths of A, b, X.*

On the other hand, we are not aware of any algorithm as in Proposition 7.5.1 that runs in polynomial-time if the dimension is fixed. This situation remains unclear even if we replace the additive term $d + 1$ in the statement by any additive term that only depends on d.

7.5.1 Computability for d=2

While we do not know any algorithm computing the relaxation complexity in general dimension, we present one that works for dimension two. The idea of our algorithm relies on the concept of a "guard set": Given a full-dimensional, polyhedral set $X \subseteq \mathbb{Z}^d$, we say that a set $G \subseteq \mathbb{Z}^d \setminus X$ is a *guard set* of X if for every integer point $p \in \mathbb{Z}^d \setminus X$ the set $\operatorname{conv}(X \cup \{p\})$ contains a point of G.

Let G be a guard set of X and let $P \subseteq \mathbb{R}^d$ be any polyhedron containing X. Then P is a relaxation of X if and only if $P \cap G = \emptyset$. Thus, the relaxation complexity of X is the smallest number of facets of any polyhedron P with $X \subseteq P$ and $P \cap G = \emptyset$. Since X is full-dimensional, this is equal to the smallest number k for which there exists a set of k

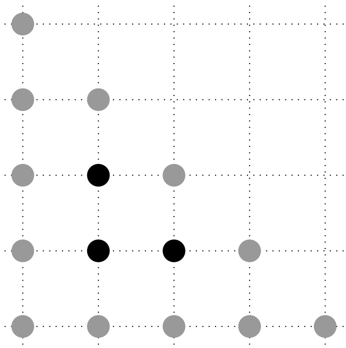

Figure 7.2: The gray points form a guard set of the black points.

linear inequalities (i) that are valid for X, and (ii) each $g \in G$ violates at least one of the inequalities. Equivalently, defining $\mathcal{S}$ to be the set of all subsets $S \subseteq G$ for which there exists a linear inequality (i') that is valid for X, and (ii') that is violated by every point of S, the relaxation complexity of X is the smallest number of sets of $\mathcal{S}$ needed to cover G. Observe that if G is finite, then the set $\mathcal{S}$ – including one corresponding linear inequality for each of its elements – can be computed by solving some auxiliary systems of linear inequalities. Thus, once we computed a smallest number k of sets in $\mathcal{S}$ to cover G, we can even compute a relaxation of X that has only k facets. We conclude:

Proposition 7.5.3. *There are algorithms for the following tasks: For a full-dimensional, polyhedral set $X \subseteq \mathbb{Z}^2$ and a guard set G of X, both given explicitly, compute*

a) the relaxation complexity of X.

b) a minimum-size relaxation of X.

Unfortunately, not every set X has a finite guard set. For instance, it is easy to see that every guard set of $\{0, 0, 0^\intercal, (1, 0, 0)^\intercal, (0, 1, 0)^\intercal, (0, 0, 1)^\intercal\}$ needs to contain the set $\{(1, 1, i)^\intercal \mid i \in \mathbb{Z}_{>0}\}$. In contrast, we will show that every two-dimensional set X has a finite guard set. To this end, we first need two basic geometric observations.

Lemma 7.5.4. *Let $t \in \mathbb{Z} \setminus \{0\}$ be some nonzero integer and $p = (p_1, p_2)^\intercal \in \mathbb{Z}^2$ some integer point with $p_2 > 0$. Then there exists an integer point $q = (q_1, 1)^\intercal \in \mathbb{Z}^2$ with $q \in \operatorname{conv}(\{(0,0)^\intercal, (t,0)^\intercal, p\})$.*

Proof. By possibly multiplying t and the first coordinate of p by -1, we may assume that t is positive. Note that for every point $q' = (x, 1)^\intercal \in \mathbb{R}^2$ we have that q' is contained in $\operatorname{conv}(\{(0,0)^\intercal, (t,0)^\intercal, p\})$ if and only if

$$\frac{p_1}{p_2} \le x \le t + \frac{p_1 - t}{p_2}$$

holds. Thus, it remains to show that the interval $[\frac{p_1}{p_2}, t + \frac{p_1 - t}{p_2}]$ contains an integer number, which is equivalent to show that $[p_1, p_2 \cdot t + p_1 - 1]$ contains an integer number $\bar{x} \in \mathbb{Z}$ that is divisible by p_2. To this end, let us choose $r \in \{0, \dots, p_2 - 1\}$ such that $\bar{x} := p_1 + r \in \mathbb{Z}$ is divisible by p_2 and confirm that

$$p_1 \le \bar{x} \le p_1 + (p_2 - 1) \le t \cdot p_2 + p_1 - 1$$

holds since $t \ge 1$. $\square$

Lemma 7.5.5. *Let $v, w, p \in \mathbb{Z}^2$ be pairwise distinct points and $a \in \mathbb{Z}^2$, $\beta \in \mathbb{Z}$ such that*

- *the entries of a are coprime,*
- *$\langle a, v \rangle = \langle a, w \rangle = \beta$, and*
- *$\langle a, p \rangle > \beta$.*

Then there exists an integer point $q \in \mathbb{Z}^2 \cap \operatorname{conv}(\{v, w, p\})$ with $\langle a, q \rangle = \beta + 1$.

Proof. Since the entries of a are coprime, there exists a unimodular matrix $C \in \mathbb{Z}^{2\times 2}$ with $Ca = (0,1)^T$. We define the affine unimodular map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ via $f(x) := C^\intercal x + v$ and set $v' := f^{-1}(v)$, $w' = f^{-1}(w)$, and $p' := f^{-1}(p)$. First, we clearly have $v' = (0,0)^\intercal$. Second, denoting the second coordinate of w' by w'_2, we obtain

$$w'_2 = \langle Ca, w' \rangle = \langle a, C^\intercal w' \rangle = \langle a, f(w') \rangle - \langle a, v \rangle = \langle a, w \rangle - \langle a, v \rangle = \beta - \beta = 0.$$

Thus, since $f^{-1}(v)$ and $f^{-1}(w)$ are distinct, there exists a nonzero integer $t \in \mathbb{Z} \setminus \{0\}$ with $w' = (t, 0)^\intercal$. Third, denoting the second coordinate of p' by p'_2, we have

$$p'_2 = \langle Ca, p' \rangle = \langle a, C^\intercal p' \rangle = \langle a, f(p') \rangle - \langle a, v \rangle = \langle a, p \rangle - \langle a, v \rangle > \beta - \beta = 0.$$

By Lemma 7.5.4, there exists an integer point $q' = (q'_1, 1)^\intercal \in \mathbb{Z}^2$ with $q' \in \operatorname{conv}(\{v', w', p'\})$. Defining $q := f(q')$, we obtain that q is an integer point that is contained in $\operatorname{conv}(\{v, w, p\})$ and that satisfies

$$\langle a, q \rangle = \langle a, C^\intercal q' + v \rangle = \langle Ca, q' \rangle + \langle a, v \rangle = 1 + \beta,$$

as desired. □

Proposition 7.5.6. *Every finite, polyhedral, full-dimensional set $X \subseteq \mathbb{Z}^2$ has a finite guard set.*

Proof. Let z be any point in the interior of $\operatorname{conv}(X)$. Let further $v^1, \ldots, v^k$ be the vertices of $\operatorname{conv}(X)$ in cyclic order and set $v^{k+1} := v^1$. For each $i \in \{1, \ldots, k\}$ we define the translated convex cone

$$K_i := \operatorname{ccone}(\{v^i - z, v^{i+1} - z\}) + z,$$

where $\operatorname{ccone}(\cdot)$ denotes the convex conic hull. For each $i \in \{1, \ldots, k\}$ there exist $a^i \in \mathbb{Z}^2$ and $\beta_i \in \mathbb{Z}$ such that

- the entries of a are coprime,
- $\langle a^i, v^i \rangle = \langle a^i, v^{i+1} \rangle = \beta_i$, and
- $\langle a^i, x \rangle \leq \beta_i$ holds for all $x \in X$.

We claim that the set

$$G := \bigcup_{i=1}^{k} \Big(K_i \cap \{q \in \mathbb{Z}^2 \mid \langle a^i, q \rangle = \beta_i + 1\} \Big)$$

is a finite guard set of X. We refer to Figure 7.5.1 for an illustration of the following argumentation. First, since z is an interior point of $\operatorname{conv}(X)$, each set $K_i \cap \{q \in \mathbb{R}^2 \mid \langle a^i, q \rangle = \beta_i + 1\}$ is bounded and hence contains finitely many integer points. Thus, G is finite. Second, since $\operatorname{conv}(X) =$

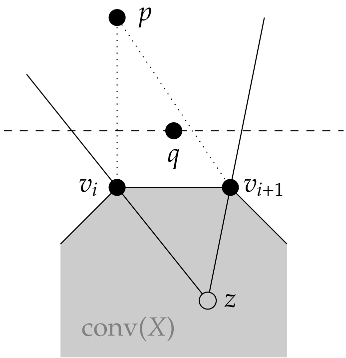

Figure 7.3: Illustration of the proof of Proposition 7.5.6.

$\{x \in \mathbb{R}^2 \mid \langle a^i, x\rangle \leq \beta_i$ for all $i = 1,\ldots,k\}$, we have $G \subseteq \mathbb{Z}^2 \setminus X$. Finally, we have to show that G is a guard set of X. To this end, let $p \in \mathbb{Z}^2 \setminus X$ be any integer point outside of X. Since we have $X = \operatorname{conv}(X) \cap \mathbb{Z}^2$ and $\cup_{i=1}^k K_i = \mathbb{R}^2$, there exists some index i with $p \in K_i$. Observe that $v := v^i$, $w := v^{i+1}$, $a := a^i$, and $\beta := \beta_i$ satisfy the requirements of Lemma 7.5.5 (note that we have $\langle a, p\rangle > \beta$ since p is not contained in $\operatorname{conv}(X)$). Thus, there exists an integer point $q \in \mathbb{Z}^2 \cap \operatorname{conv}(\{v^i, v^{i+1}, p\})$ with $\langle a^i, q\rangle = \beta_i + 1$. By $\operatorname{conv}(\{v^i, v^{i+1}, p\}) \subseteq K_i$, the point q is contained in G and we obtain

$$q \in G \cap \operatorname{conv}(\{v^i, v^{i+1}, p\}) \subseteq G \cap \operatorname{conv}(X \cup \{p\}),$$

as claimed. □

Given a two-dimensional polyhedral set $X \subseteq \mathbb{Z}^2$, the proof of Proposition 7.5.6 gives an explicit construction of a finite guard G set of X. It is straight-forward to verify that all necessary steps to compute G can be performed by a deterministic algorithm. Furthermore, note that for every finite polyhedral set $X \subseteq \mathbb{Z}^2$ with $\dim(X) < 2$, we have $\operatorname{rc}(X) = 2$ if X contains at least two points and $\operatorname{rc}(X) = 0$, otherwise. We can summarize:

Theorem 7.5.7. *There are algorithms for the following tasks: Given a polyhedral set $X \subseteq \mathbb{Z}^2$, compute* *a) a finite guard set of X.*

b) the relaxation complexity of X.

c) a minimum-size relaxation of X.

8

Describing Integer Points in Polytopes: Obstructions

8.1 Examples

We start this part by providing lower bounds on the relaxation complexities of two specific choices for X, namely $X = \{0,1\}^d$ being the set of vertices of the hypercube, and $X = \Delta_d$ the set of vertices of the standard simplex (see Section 7.3.2). By Lemma 7.2.1 and the fact that $\operatorname{conv}(\Delta_d)$ has only $d+1$ facets, we have $\operatorname{rc}(X) \leq d+1$ for both of these sets. On the other side, as already discussed in Section 7.3.2, every *rational* relaxation for one of these sets must be bounded. While this statement turned out to be wrong for arbitrary relaxations of Δ_d, we show that it can be transferred to arbitrary relaxations of $\{0,1\}^d$, implying $\operatorname{rc}(\{0,1\}^d) = d+1$. For the case of $X = \Delta_d$, we are able to give at least some lower bound on its relaxation complexity that grows in d.

8.1.1 Hypercube

Recall that every polyhedron containing $\{0,1\}^d$ (and hence being d-dimensional) that has less than $d+1$ facets must be unbounded. In what follows, we will show that every such (possibly irrational) polyhedron must contain infinitely many integer points, and hence cannot be a relaxation of $\{0,1\}^d$. For this purpose, we make use of Minkowski's classical theorem:

Theorem 8.1.1 (Minkowski [59]). *Any convex set that is symmetric with respect to the origin and with volume greater than 2^d contains a nonzero integer point.*

For $\varepsilon > 0$ let $B_\varepsilon := \{x \in \mathbb{R}^d \mid \|x\|_2 < \varepsilon\}$, where $\|\cdot\|_2$ denotes the Euclidean norm, be the open ball with radius ε. As a direct consequence of Minkowski's theorem, the following Lemma is useful for our argumentation.

Lemma 8.1.2. *Let $c \in \mathbb{R}^d \setminus \{\mathbb{O}\}$, $\lambda_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then the set*

$$L(c, \lambda_0, \varepsilon) := \left\{\lambda c \in \mathbb{R}^d \mid \lambda \geq \lambda_0\right\} + B_\varepsilon$$

contains infinitely many integer points.

Proof. Let us define $L(c, \varepsilon) := \{\lambda c \in \mathbb{R}^d \mid \lambda \in \mathbb{R}\} + B_\varepsilon$, which is convex, symmetric with respect to the origin, and has an infinite volume. We first argue that $L(c, \varepsilon)$ contains infinitely many integer points. Clearly, this is true if c is a multiple of a rational vector. Thus, let us assume that c is not a multiple of a rational vector, which means that we have

$$\{\lambda c \mid \lambda \in \mathbb{R}\} \cap \mathbb{Z}^d = \{\mathbb{O}\}. \tag{8.1}$$

Setting $\varepsilon_1 := \varepsilon$, by Theorem 8.1.1, $L(c, \varepsilon_1)$ contains a point $p_1 \in \mathbb{Z}^d \setminus \{\mathbb{O}\}$. By (8.1) there exists some $\varepsilon_2 > 0$ such that $L(c, \varepsilon_2) \subseteq L(c, \varepsilon_1)$ and $p_1 \notin L(c, \varepsilon_2)$. Again, by Theorem 8.1.1, $L(c, \varepsilon_2)$ also contains a point $p_2 \in \mathbb{Z}^d \setminus \{\mathbb{O}\}$. Further, there is again some $\varepsilon_3 > 0$ such that $L(c, \varepsilon_3) \subseteq L(c, \varepsilon_2)$ and $p_2 \notin L(c, \varepsilon_3)$. Iterating these arguments, we obtain an infinite sequence (ε_i, p_i) such that $p_i \in L(c, \varepsilon_i) \cap \mathbb{Z}^d \subseteq L(c, \varepsilon) \cap \mathbb{Z}^d$ and $p_i \notin L(c, \varepsilon_{i+1})$ for all i. In particular, all p_i are distinct, and hence $L(c, \varepsilon)$ indeed contains infinitely many integer points.

As $L(c, \varepsilon)$ contains infinitely many integer points and is symmetric with respect to the origin, we obtain that $L(c, 0, \varepsilon)$ also contains infinitely many integer points. Let us set $\bar{\lambda} := \max\{0, \lambda_0\}$ and observe that $L(c, 0, \varepsilon) \setminus L(c, \bar{\lambda}, \varepsilon)$ is bounded. Thus, $L(c, \bar{\lambda}, \varepsilon)$ contains infinitely many integer points and so does $L(c, \lambda_0, \varepsilon)$ since we have $L(c, \bar{\lambda}, \varepsilon) \subseteq L(c, \lambda_0, \varepsilon)$. □

We are ready to rule out the existence of relaxations of $\{0,1\}^d$ that have less than $d+1$ facets.

Theorem 8.1.3. *For $d \geq 1$, we have that* $\mathrm{rc}(\{0,1\}^d) = d+1$ *holds.*

Proof. By Lemma 7.2.1 and the previous discussion, it remains to show that every unbounded polyhedron $R \subseteq \mathbb{R}^d$ with $\{0,1\}^d \subseteq R$ contains infinitely many integer points, which we will prove by induction over $d \geq 1$. Clearly, the claim is true for $d = 1$. For $d \geq 2$, let $c \in \mathbb{R}^d \setminus \{\mathbb{O}\}$ be a direction such that $x + \lambda c$ is contained in R for every $x \in R$, $\lambda \geq 0$. Since $\{0,1\}^d$ is invariant under affine maps that map a subset of coordinates x_i to $1 - x_i$, we may assume that every entry of c is nonnegative.

If every coordinate of c is strictly positive, then there is some $\lambda_0 > 0$ such that $\lambda_0 c$ is in the interior of $[0,1]^d$. Thus, there exists some $\varepsilon > 0$ such that $\lambda_0 c + B_\varepsilon \subseteq [0,1]^d \subseteq R$. By the definition of c and ε, we thus obtained $L(c, \lambda_0, \varepsilon) \subseteq R$. By Lemma 8.1.2, it follows that $L(c, \lambda_0)$ contains infinitely many integer points and so does R.

Otherwise, we may assume that $c_d = 0$ holds. Let $H := \{x \in \mathbb{R}^d \mid x_d = 0\}$ and $p \colon H \to \mathbb{R}^{d-1}$ be the projection onto the first $d-1$ coordinates. Then, the polyhedron $R' = p(R \cap H)$ is still unbounded and contains $\{0,1\}^{d-1} = p(\{0,1\}^d)$. By the induction hypothesis, R' contains infinitely many integer points and so does R. □

8.1.2 Simplex

In the previous section, we showed that every relaxation of $X = \{0,1\}^d$ must be bounded, which is not the case for $X = \Delta_d$ (see Section 7.3.2). Although we believe that every unbounded relaxation of Δ_d has at least $d+1$ facets (which would imply $\mathrm{rc}(\Delta_d) = d+1$), we can only give a much weaker bound on the number of facets of such relaxations.

Proposition 8.1.4. *For every $k \geq 1$, we have* $\mathrm{rc}(\Delta_{k!}) \geq k$.

Proof. Clearly, the claim is true for $k = 1$. Further, it is easy to see that $\mathrm{rc}(\Delta_m) \leq \mathrm{rc}(\Delta_n)$ holds for all $m \leq n$. Thus, by setting $d(k) := k! - 1$, it suffices to show that $\mathrm{rc}(\Delta_{d(k)}) \geq k$ holds for all $k \geq 2$, which we will show by induction over k. Note that the latter statement is true for $k = 2$. Let us assume that it is wrong for some $k \geq 3$, i.e., there exists a relaxation $R \subseteq \mathbb{R}^d$ of $\Delta_{d(k)}$ that has $\ell < k$ facets. Since $\ell < k \leq d(k) = \dim(R)$, R has to be unbounded.

We claim that every integer point p of $\Delta_{d(k)}$ must lie in at least one facet of R. Otherwise, since R is unbounded, there exist some $\varepsilon > 0$ and $c \in \mathbb{R}^d \setminus \{\mathbb{O}\}$ such that $p + L(c, 0, \varepsilon)$ is contained in R. By Lemma 8.1.2, $L(c, 0, \varepsilon)$ contains infinitely many integer points and so does R, a contradiction.

Hence, there must be a facet of R that contains $t \geq \frac{d(k)+1}{\ell}$ vertices $v_1, \dots, v_t$ of $\Delta_{d(k)}$. Let H be the affine subspace spanned by $v_1, \dots, v_t$ and let $\varphi \colon H \cap \mathbb{Z}^{d(k)} \to \mathbb{Z}^{t-1}$ be an affine isomorphism mapping $\{v_1, \dots, v_t\}$ to Δ_{t-1}. Extending φ to an affine map from H to $\mathbb{R}^{t-1}$ yields that $R' := \varphi(R \cap H)$ is a relaxation of Δ_{t-1}. Since we have

$$t - 1 \geq \frac{d(k) + 1}{\ell} - 1 \geq \frac{d(k) + 1}{k} - 1 = \frac{k! - 1 + 1}{k} - 1 = d(k - 1),$$

the induction hypothesis implies that R' has at least $k - 1$ facets. On the other hand, note that R' has at most $\ell - 1$ facets. This implies $k \leq \ell$, a contradiction to our assumption. □

8.2 General Lower Bounds

The lower bounds on the relaxation complexity we have seen so far were based on arguments very specific to the particular choice of X. Furthermore, the techniques we used so far can only yield bounds on $\mathrm{rc}(X)$ that are at most $\dim(X) + 1$. However, one main purpose of this chapter is to show that the relaxation complexities of some sets X grow exponentially in the dimension of X. In this section, we will develop a simple technique that allows to obtain such strong bounds in many relevant cases.

8.2.1 Random 0/1-Sets

Most sets we consider in the remainder of this chapter only contain binary points. Before we derive bounds on the relaxation complexities of specific choices of such sets, we would like to understand the asymptotic behavior of the relaxation complexity of a generic, say random, set $X \subseteq \{0, 1\}^d$. First, we give a bound the number of subsets of $\{0, 1\}^d$ with bounded relaxation complexity. Our argumentation is based on simple counting arguments, which, for instance, have been used in [71].

Lemma 8.2.1. *There exists a constant $c \in \mathbb{R}$ such that*

$$\left|\left\{X \subseteq \{0,1\}^d \mid \mathrm{rc}(X) \leq k\right\}\right| \leq 2^{k \cdot cd^3}$$

holds for every integers $d, k \geq 1$.

Proof. In Section 7.3 we have shown that there exists a constant $c' \in \mathbb{R}$ such that to each set $X \subseteq \{0,1\}^d$ we can assign a matrix $A \in \mathbb{Q}^{\mathrm{rc}(X) \times d}$ and a vector $b \in \mathbb{Q}^{\mathrm{rc}(X)}$ with (i) $X = \{x \in \{0,1\}^d \mid Ax \leq b\}$, and (ii) each entry in A, b has encoding length bounded by $c' \cdot d^2$ (see Proposition 7.3.1 and the discussion of Corollary 7.3.2). This means that by (ii) the number of possible choices for each entry in A, b is at most $2^{c' \cdot d^2}$. Thus, the total number of such pairs (A, b) that can be assigned to some set $X \subseteq \{0,1\}^d$ with $\mathrm{rc}(X) \leq k$ is at most $(2^{c' \cdot d^2})^{k \cdot d + d}$. Setting $c := c' + 1$, we obtain the claim by observing that the assignment is injective by (i). □

In Section 7.4.2 we have seen that the relaxation complexity of every set $X \subseteq \{0,1\}^d$ with $d \geq 4$ is at most 2^d, see Inequality 7.12. Together with the above lemma, we now obtain that the relative amount of sets having a relaxation complexity that differs from this upper bound by only a polynomial factor is doubly-exponentially small:

Proposition 8.2.2. *There exists a constant $c \in \mathbb{R}$ such that if we pick a set $X \subseteq \{0,1\}^d$ uniformly at random (for fixed d), then we have*

$$\mathbb{P}\left[\mathrm{rc}(X) \leq \tfrac{2^d}{c \cdot d^3}\right] \leq \frac{1}{2^{2^{d-1}}}.$$

Proof. Note that the total number of subsets of $\{0,1\}^d$ is 2^{2^d}. Thus, by Lemma 8.2.1 there exists some constant $c' \in \mathbb{R}$ such that

$$\mathbb{P}[\mathrm{rc}(X) \leq k] \leq 2^{k \cdot c' d^3 - 2^d}$$

holds and hence setting $c := \frac{1}{2}c'$ and $k := \frac{2^d}{c \cdot d^3}$ yields the claim. □

In particular, we have shown the *existence* of families of subsets of binary points whose extension complexities grow exponentially in the dimension. In Section 8.3 we will visit several specific examples of this type.

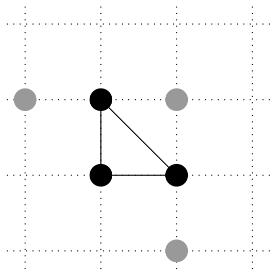

Figure 8.1: Hiding set (gray) for the vertices of the standard 2-simplex (black).

8.2.2 Hiding Sets

We now provide a simple framework to obtain lower bounds on the relaxation complexity of specific sets. To this end, let $X \subseteq \mathbb{Z}^d$ be some polyhedral set and consider a set $H \subseteq \operatorname{aff}(X) \cap \mathbb{Z}^d \setminus X$, where $\operatorname{aff}(\cdot)$ denotes the affine hull. We call H a *hiding set* for X if we have $\operatorname{conv}\{a, b\} \cap \operatorname{conv}(X) \neq \emptyset$ for any two distinct points $a, b \in H$. Suppose $R \subseteq \mathbb{R}^d$ is any relaxation for X. Since we have $H \subseteq \operatorname{aff}(X) \subseteq \operatorname{aff}(R)$, every point of H must be separated from X by some facet-defining inequality of R. Suppose that a linear inequality $\langle \alpha, x \rangle \leq \beta$ that is valid for R is violated by two distinct points $a, b \in H$. Since H is a hiding set, there exists a point $\bar{x} \in \operatorname{conv}\{a, b\} \cap \operatorname{conv}(X)$. Clearly, we have that $\bar{x}$ also violates $\langle \alpha, x \rangle \leq \beta$, which is a contradiction since the latter is valid for $\operatorname{conv}(X) \subseteq R$. Thus, every facet-defining inequality of R is violated by at most one point in H and hence R has at least $|H|$ facets. We conclude:

Proposition 8.2.3. *Let $X \subseteq \mathbb{Z}^d$ be polyhedral and $H \subseteq \operatorname{aff}(X) \cap \mathbb{Z}^d \setminus X$ a hiding set for X. Then we have* $\operatorname{rc}(X) \geq |H|$.

In the next section, we will demonstrate that this simple concept is a powerful tool to provide exponential lower bounds on the relaxation complexities of numerous interesting sets X. However, let us first illustrate that the hiding-set bound has its limitations. Consider the set $\Delta_d \subseteq \{0, 1\}^d$ consisting of the vertices of the standard d-simplex. As an example, see Figure 8.1 illustrating a hiding set for Δ_2 that has cardinality three, which yields a simple proof of the fact that any relaxation for these points must have at least three facets. Unfortunately, it turns out that we cannot construct larger hiding sets for any Δ_d. Recall that the relaxation complexity of Δ_d gets arbitrarily large (for large enough d), see Proposition 8.1.4.

Proposition 8.2.4. *Every hiding set for Δ_d has cardinality at most three.*

Proof. Let H be any hiding set for Δ_d. Then, for each of the inequalities $x_i \geq 0$ for $i = 1, \dots, d$ as well as $\sum_{i=1}^{d} x_i \leq 1$, there exists at most one point in H violating it. In particular, at most one of the points in H is contained in the nonnegative orthant.

For the sake of contradiction, let us assume that H contains at least four elements. Then there are distinct points $a, b, p, q \in H$ with $a_i < 0$ and $b_j < 0$ for some $i, j \in \{1, \dots, d\}$ with $i \neq j$. Since $\lambda a + (1 - \lambda)p \in \Delta_d \subseteq \mathbb{R}^d_{\geq 0}$ for some $\lambda \in (0, 1)$, we must have $p_i > 0$. As p_i is an integer, it follows that $p_i \geq 1$ holds. Analogously, we obtain $p_j, q_i, q_j \geq 1$.

Now consider any point $y = \lambda p + (1 - \lambda)q \in \operatorname{conv}\{p, q\}$ with $\lambda \in [0, 1]$. Note that every point $x \in \operatorname{conv} \Delta_d$ satisfies $x_i + x_j \leq 1$. But since $p_i, p_j, q_i, q_j \geq 1$, we obtain $y_i + y_j \geq 2$ and hence $y \notin \operatorname{conv} \Delta_d$. Thus, we have $\operatorname{conv}\{p, q\} \cap \operatorname{conv} \Delta_d = \emptyset$, a contradiction to H being a hiding set for Δ_d. $\square$

8.3 Exponential Lower Bounds for some Specific Structures

In this section, we provide lower bounds on the relaxation complexities of several sets $X \subseteq \mathbb{Z}^d$ that one encounters frequently in combinatorial optimization. By dividing these sets into three classes, we try to identify general structures that are hard to model in the context of relaxations.

8.3.1 Connectivity and Acyclicity

In many integer-programming formulations for practical applications, the feasible solutions are subsets of edges of graphs that are required to form connected or acyclic subgraphs. Quite often in these cases, there exist well-known polynomial size integer-programming formulations that use auxiliary variables. For instance, for the *spanning tree polytope* there are even polynomial-size extended formulations (see Section 3.2) that can be easily adapted to also work for the *connector polytope* $\operatorname{conv}(X_{\text{conn}}(n))$ (see below). In contrast, we give exponential lower bounds on the relaxation complexities of some important representatives of this structural class.

STSP and ATSP

As a first application of the hiding-set bound, we will show that the subtour polytope has asymptotically smallest size (in the exponential sense) among all relaxations of $X_{\text{STSP}}(n)$, i.e., that $\operatorname{rc}(X_{\text{STSP}}(n)) = 2^{\Theta(n)}$ holds. In fact, we will also give an exponential lower bound for the directed version $X_{\text{ATSP}}(n) \subseteq \{0,1\}^{A_n}$, which is the set of characteristic vectors of directed hamiltonian cycles in the complete directed graph on n nodes whose arcs we denote by A_n. We will first construct a large hiding set for $X_{\text{ATSP}}(n)$. Towards this end, let $n = 2(N+1)$ for some integer $N \geq 0$ and let us consider the complete directed graph on the node set

$$V := \{v_1, \ldots, v_{N+1}, w_1, \ldots, w_{N+1}\}$$

of cardinality n. For a binary vector $b \in \{0,1\}^N$ let us further define the arc set

$$\begin{aligned}\mathcal{A}_b := &\Big\{(v_{N+1}, v_1),\ (w_{N+1}, w_1)\Big\} \\ &\cup \bigcup_{i : b_i = 0} \Big\{(v_i, v_{i+1}),\ (w_i, w_{i+1})\Big\} \ \cup \bigcup_{i : b_i = 1} \Big\{(v_i, w_{i+1}),\ (w_i, v_{i+1})\Big\},\end{aligned}$$

see Figure 8.2 for an example. Note that $\mathcal{A}_b$ is a directed hamiltonian cycle on the node set V if and only if $\sum_{i=1}^N b_i$ is odd. Thus, the set

$$H_N := \left\{\chi(\mathcal{A}_b) \mid b \in \{0,1\}^N,\ \sum_{i=1}^N b_i \text{ is even}\right\}$$

is clearly disjoint from $X_{\text{ATSP}}(2(N+1))$. Here, we will only consider graphs on $2(N+1)$ nodes. It is easy to transfer the following observations to complete graphs with an odd number of nodes by replacing arc (v_{N+1}, v_1) in $\mathcal{A}_b$ by a directed path including one additional node.

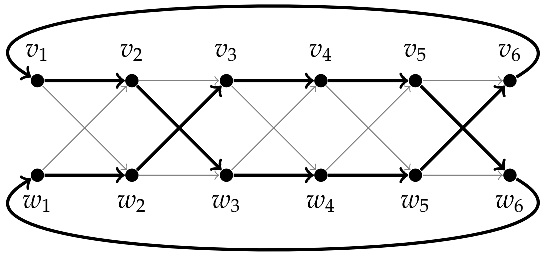

Figure 8.2: Construction of the set $\mathcal{A}_b$ for $b = (0, 1, 0, 0, 1)$.

Lemma 8.3.1. *H_N is a hiding set for $X_{\mathrm{ATSP}}(2(N+1))$.*

Proof. First, note that

$$H_N \subseteq \operatorname{aff}(X_{\mathrm{ATSP}}(2(N+1))) = \left\{x \in \mathbb{R}^{A_n} \mid x(\delta^{\mathrm{in}}(v)) = x(\delta^{\mathrm{out}}(v)) = 1 \ \forall\, v \in V\right\}$$

holds. Let $b, b' \in \{0,1\}^N$ be distinct with $\sum_{i=1}^N b_i$ even and $\sum_{i=1}^N b'_i$ even. Then there exists an index $j \in \{1, \dots, N\}$ with $b_j \neq b'_j$. Consider the binary vectors $c, c' \in \{0,1\}^N$, where c arises from b by replacing the j-th component of b by $1 - b_j$, and c' arises from b' by replacing the j-th component of b' by $1 - b'_j$. Clearly, we have that $\sum_{i=1}^N c_i$ is odd and $\sum_{i=1}^N c'_i$ is odd and hence $\chi(\mathcal{A}_c)$ and $\chi(\mathcal{A}_{c'})$ are both contained in $X_{\mathrm{ATSP}}(2(N+1))$. Finally, note that that we have

$$\chi(\mathcal{A}_b) + \chi(\mathcal{A}_{b'}) = \chi(\mathcal{A}_c) + \chi(\mathcal{A}_{c'})$$

and hence $\operatorname{conv}(\{\chi(\mathcal{A}_b), \chi(\mathcal{A}_{b'})\}) \cap \operatorname{conv}(X_{\mathrm{ATSP}}(2(N+1))) \neq \emptyset$, as required. □

Theorem 8.3.2. *The asymptotic growth of $\operatorname{rc}(X_{\mathrm{ATSP}}(n))$ and $\operatorname{rc}(X_{\mathrm{STSP}}(n))$ is $2^{\Theta(n)}$.*

Proof. By Lemma 8.3.1 and Proposition 8.2.3, we obtain $\operatorname{rc}(X_{\mathrm{ATSP}}(n)) \leq |H_N| = 2^{\Omega(n)}$. Furthermore, note that one can construct a relaxation of $X_{\mathrm{ATSP}}(n)$ that is a variant of the formulation in (6.2) and consists of $2^{\Theta(n)}$ linear inequalities, implying $\mathrm{rc}(X_{\mathrm{ATSP}}(n)) = 2^{\Theta(n)}$. By replacing all directed arcs with their undirected versions, the same statement can be obtained analogously for the case of $X_{\mathrm{STSP}}(n)$. □

Connected Sets

Let $X_{\mathrm{conn}}(n)$ be the set of all characteristic vectors of edge sets that form a connected spanning subgraph in the complete graph (V_n, E_n) on n nodes. The polytope

$$\left\{x \in [0,1]^{E_n} \mid x(\delta(S)) \geq 1 \;\forall\, \emptyset \neq S \subsetneq V_n\right\}$$

is a relaxation for $X_{\mathrm{conn}}(n)$. Thus, we have that $\mathrm{rc}(X_{\mathrm{conn}}(n)) \leq \mathcal{O}(2^n)$ holds.

For a lower bound, consider again the undirected version of our set H_N. Since each point in H_N belongs to a node-disjoint union of two cycles, we have $H_N \cap X_{\mathrm{conn}}(n) = \emptyset$. Further, we know that for any $a, b \in H_N$

$$\emptyset \neq \mathrm{conv}\{a,b\} \cap \mathrm{conv}(X_{\mathrm{STSP}}(n)) \subseteq \mathrm{conv}\{a,b\} \cap \mathrm{conv}(X_{\mathrm{conn}}(n))$$

holds. Together with $H_N \subseteq \mathrm{aff}(X_{\mathrm{conn}}(n)) = \mathbb{R}^{E_n}$ this implies that H_N is also a hiding set for $X_{\mathrm{conn}}(n)$. We obtain:

Corollary 8.3.3. *The asymptotic growth of* $\mathrm{rc}(X_{\mathrm{conn}}(n))$ *is* $2^{\Theta(n)}$.

Branchings and Forests

Besides connectivity, we show that, in general, it is also hard to force acyclicity in the context of relaxations. To this end, let $X_{\mathrm{arb}}(n)$ ($X_{\mathrm{sp.trees}}(n)$) be the set of characteristic vectors of arborescences (spanning trees) in the complete directed (undirected) graph on n nodes.

Theorem 8.3.4. *The asymptotic growth of* $\mathrm{rc}(X_{\mathrm{arb}}(n))$ *and* $\mathrm{rc}(X_{\mathrm{sp.trees}}(n))$ *is* $2^{\Theta(n)}$.

Proof. First, we remark that both the arborescence polytope and the spanning-tree polytope have $\mathcal{O}(2^n)$ facets, see [74, Cor. 52.6c & Cor. 50.7c]. Thus, we have an upper bound of $\mathcal{O}(2^n)$ for both $\mathrm{rc}(X_{\mathrm{arb}}(n))$ and $\mathrm{rc}(X_{\mathrm{sp.trees}}(n))$. For a lower bound, let us modify the definition of $\mathcal{A}_b$ by removing arc (w_{N+1}, w_1). Then, for every $b \in \{0,1\}^N$ with $\sum_{i=1}^N b_i$ even, we have that $\mathcal{A}_b$ is a node-disjoint union of a cycle and a path and hence not an arborescence. By following the proof of Lemma 8.3.1, we still have

$$\chi(\mathcal{A}_b) + \chi(\mathcal{A}_{b'}) = \chi(\mathcal{A}_c) + \chi(\mathcal{A}_{c'}),$$

where $\mathcal{A}_c$ and $\mathcal{A}_{c'}$ are spanning arborescences. (Actually, they are in fact directed paths visiting each node.) Since $\mathrm{aff}(X_{\mathrm{arb}}(n)) = \mathbb{R}^{A_n}$, we therefore obtain that the modified set H_N is a hiding set for $X_{\mathrm{arb}}(n)$. By undirecting all arcs, H_N also yields a hiding set for $X_{\mathrm{sp.trees}}(n)$. Again, by Proposition 8.2.3, we deduce a lower bound of $|H_N| = 2^{\Theta(n)}$ for both $\mathrm{rc}(X_{\mathrm{arb}}(n))$ and $\mathrm{rc}(X_{\mathrm{sp.trees}}(n))$. □

Let $X_{\mathrm{branch}}(n)$ ($X_{\mathrm{forests}}(n)$) be the set of characteristic vectors of branchings (forests) in the complete directed (undirected) graph on n nodes.

Corollary 8.3.5. *The asymptotic growth of* $\mathrm{rc}(X_{\mathrm{branch}}(n))$ *and* $\mathrm{rc}(X_{\mathrm{forests}}(n))$ *is* $2^{\Theta(n)}$.

Proof. The claim follows from Theorem 8.3.4 and the facts

$$X_{\mathrm{arb}}(n) = X_{\mathrm{branch}}(n) \cap \Big\{x \in \mathbb{R}^{A_n} : \sum_{a \in \mathbb{R}^{A_n}} x_a = n-1\Big\},$$
$$X_{\mathrm{sp.trees}}(n) = X_{\mathrm{forests}}(n) \cap \Big\{x \in \mathbb{R}^{E_n} : \sum_{e \in \mathbb{R}^{E_n}} x_e = n-1\Big\}.$$

□

8.3.2 Distinctness

Another common component of practical integer-programming formulations is the requirement of distinctness of a certain set of vectors or variables. Here, we consider two general cases in which we can also show that the benefit of auxiliary variables is essential.

Binary All-Different

In the case of the *binary all-different* constraint, one requires the distinctness of rows of a binary matrix with m rows and n columns. The set of feasible points is therefore defined by

$$X_{\mathrm{diff}}(m,n) := \{x \in \{0,1\}^{m \times n} \mid x \text{ has pairwise distinct rows}\}.$$

As an example, [55] give integer-programming formulations to solve the coloring problem in which encode the numbers of the color classes assigned to each node by their binary representation. As a consequence, in their approach certain sets of encoding vectors have to be distinct. By separating each possible pair of equal rows by one inequality, it is further easy to give a relaxation for $X_{\text{diff}}(m,n)$ that has at most $\binom{m}{2}2^n + 2mn$ facets. In the case of $m = 2$, for instance, this bound turns out to be almost tight:

Theorem 8.3.6. *For all $n \geq 1$, we have that $\mathrm{rc}(X_{\text{diff}}(2,n)) \geq 2^n$ holds.*

Proof. Let us consider the set

$$H_{2,n} := \left\{(x,x)^\intercal \in \{0,1\}^{2\times n} \mid x \in \{0,1\}^n\right\}.$$

For $x, y \in \{0,1\}^n$ distinct, we obviously have

$$\tfrac{1}{2}\left((x,x)^\intercal + (y,y)^\intercal\right) = \tfrac{1}{2}\left((x,y)^\intercal + (y,x)^\intercal\right) \in \mathrm{conv}(X_{\text{diff}}(2,n)).$$

Since $H_{2,n} \cap X_{\text{diff}}(2,n) = \emptyset$ and $H_{2,n} \subseteq \mathrm{aff}(X_{\text{diff}}(2,n)) = \mathbb{R}^{2\times n}$, the set $H_{2,n}$ is a hiding set for $X_{\text{diff}}(2,n)$ and by Proposition 8.2.3 this implies $\mathrm{rc}(X_{\text{diff}}(2,n)) \geq |H_{2,n}| = 2^n$. □

Permutahedron

As a case in which one does not require the distinctness of binary vectors but of a set of numbers let us consider the set

$$X_{\text{perm}}(n) := \{(\pi(1),\ldots,\pi(n)) \in \mathbb{Z}^n \mid \pi \in \mathcal{S}_n\},$$

where $\mathcal{S}_n$ denotes the set of all permutations on $\{1,\ldots,n\}$. This set is the vertex set of the *permutahedron* $\mathrm{conv}(X_{\text{perm}}(n))$, which can be described via

$$\begin{aligned}\mathrm{conv}(X_{\text{perm}}(n)) = \Bigg\{x \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i &= \frac{n(n+1)}{2}\\ \sum_{i\in S} x_i &\geq \frac{|S|(|S|+1)}{2} \ \forall\, \emptyset \neq S \subset \{1,\ldots,n\}\Bigg\},\end{aligned} \tag{8.2}$$

as shown by Rado [69]. Thus, the permutahedron has $\mathcal{O}(2^n)$ facets. Furthermore, it is a good example for a polytope having many different, polynomial-size extended formulations, see, e.g., [35]. However, we show that the relaxation complexity of $X_{\text{perm}}(n)$ has exponential growth in n, which has been already observed by Eisenschmidt [25].

Theorem 8.3.7. *The asymptotic growth of* $\operatorname{rc}(X_{\text{perm}}(n))$ *is* $2^{\Theta(n)}$.

Proof. Let $m := \lfloor \frac{n}{2} \rfloor$. For every set $S \subseteq \{1, \dots, n\}$ with $|S| = m$ select an integer vector $x^S \in \mathbb{Z}^n$ with

- $\{x^S_i \mid i \in S\} = \{1, \dots, m-1\}$,
- the value $m-1$ occurs twice among the x^S_i $(i \in S)$,
- $\{x^S_i \mid i \in \{1, \dots\} \setminus S\} = \{m+2, \dots, n\}$, and
- the value $m+2$ occurs twice among the x^S_i $(i \in \{1, \dots, n\} \setminus S)$.

Note that such a vector is not contained in $\operatorname{conv}(X_{\text{perm}}(n))$ as we have

$$\sum_{i \in S} x^S_i = 1 + 2 + \dots + (|S|-1) + (|S|-1) < \frac{|S|(|S|+1)}{2}.$$

On the other hand, note that this is the only constraint in(8.2) that is violated by x^S. In particular, x^S is contained in $\operatorname{aff}(X_{\text{perm}}(n)$.

Let $S_1, S_2 \subseteq \{1, \dots, n\}$ with $|S_1| = |S_2| = m$ be distinct. We will show that $x := \frac{1}{2}(x^{S_1} + x^{S_2})$ is contained in $\operatorname{conv}(X_{\text{perm}}(n))$. Since x satisfies all constraints that are satisfied by both x^{S_1} and x^{S_2}, it remains to show that $\sum_{i \in T} x_i \geq \frac{|T|(|T|+1)}{2}$ holds for $T \in \{S_1, S_2\}$. We may assume $T = S_1$ and obtain

$$\begin{aligned}
\sum_{i \in S_1} x_i &= \frac{1}{2} \sum_{i \in S_1} x^{S_1}_i + \frac{1}{2} \sum_{i \in S_1} x^{S_2}_i \\
&= \frac{1}{2} \left(\frac{m(m+1)}{2} - 1 \right) + \frac{1}{2} \sum_{i \in S_1} x^{S_2}_i \\
&\geq \frac{1}{2} \left(\frac{m(m+1)}{2} - 1 \right) + \frac{1}{2} \left(\frac{m(m+1)}{2} + 2 \right) \\
&= \frac{m(m+1)}{2} + \frac{1}{2} \geq \frac{|T|(|T|+1)}{2}.
\end{aligned}$$

Thus, the set $H := \left\{x^S \mid S \subseteq \{1,\dots,n\},\ |S| = m\right\}$ is a hiding set for $X_{\text{perm}}(n)$. Our claim follows from Proposition 8.2.3 and the fact

$$|H| = \binom{n}{\lfloor \frac{n}{2} \rfloor} = 2^{\Theta(n)}.$$

□

8.3.3 Parity

The final structural class we consider deals with the restriction that the number of selected elements of a given set has a certain parity. Let us call a binary vector $a \in \{0,1\}^d$ to be *even* (*odd*) if the sum of its entries is even (odd). In [41] it is shown that the number of inequalities needed to separate

$$X_{\text{even}}(n) := \{x \in \{0,1\}^n \mid x \text{ is even}\}$$

from all remaining points in $\{0,1\}^n$ is exactly 2^{n-1}. This is done by showing that

$$X_{\text{odd}}(n) := \{x \in \{0,1\}^n \mid x \text{ is odd}\}$$

is a hiding set for $X_{\text{even}}(n)$ (although the notion in [41] is different from ours). Hence, with Theorem 7.4.5, we obtain:

Theorem 8.3.8. *The asymptotic growth of* $\operatorname{rc}(X_{\text{even}}(n))$ *is* $\Theta(2^n)$.

T-joins

As a well-known representative of this structural class let us consider the set $X_{T\text{-joins}}(n)$, which is, for some fixed set $T \subseteq V_n$, defined as the set of characteristic vectors of T-joins in the complete graph on n nodes. Recall that a T-join is a set $J \subseteq E_n$ of edges such that T is equal to the set of nodes of odd degree in the graph (V_n, J). Note, that if a T-join exists, then $|T|$ is even.

Theorem 8.3.9. *Let n be even and $T \subseteq V_n$ with $|T|$ even. Then the asymptotic growth of* $\operatorname{rc}(X_{T\text{-joins}}(n))$ *is at least* $2^{\Omega(n)}$.

Proof. Since n is even and $|T|$ is even, we may partition V_n into four pairwise disjoint sets T_1, T_2, U_1, U_2 with $T = T_1 \cup T_2$, $k = |T_1| = |T_2|$ and $\ell = |U_1| = |U_2|$. Let $M_1, \dots, M_k$ be pairwise edge-disjoint matchings of cardinality k that connect nodes from T_1 with nodes from T_2. Analogously, let $N_1, \ldots, N_\ell$ be pairwise edge-disjoint matchings of cardinality ℓ that connect nodes from U_1 with nodes from U_2. For $b \in \{0,1\}^k$ and $c \in \{0,1\}^\ell$ let us further define

$$J(b,c) := \Big(\bigcup_{i:b_i=1} M_i\Big) \cup \Big(\bigcup_{j:c_j=1} N_j\Big) \subseteq E_n.$$

By definition, the set $J(b,c)$ is a T-join if and only if b is odd and c is even. Let $b^* \in \{0,1\}^k$ odd and $c^* \in \{0,1\}^\ell$ even be arbitrarily chosen but fixed. Since $X_{\mathrm{odd}}(n)$ is a hiding set for $X_{\mathrm{even}}(n)$ and vice versa, it is now straight-forward to check that both sets

$$\begin{aligned} H_1 &:= \Big\{J(b,c^*) \mid b \in \{0,1\}^k \text{ even}\Big\}, \\ H_2 &:= \Big\{J(b^*,c) \mid c \in \{0,1\}^\ell \text{ odd}\Big\} \end{aligned}$$

are hiding sets for $X_{T-\mathrm{joins}}(n)$. Our claim follows from Proposition 8.2.3 and the fact

$$\max\{|H_1|,|H_2|\} = \max\Big\{2^{k-1}, 2^{\ell-1}\Big\} = \max\Big\{2^{\frac{1}{2}|T|-1}, 2^{\frac{1}{2}(n-|T|)-1}\Big\} \geq 2^{\frac{1}{2}\cdot\frac{n}{2}-1}.$$

□

9

Describing Integer Points in Polytopes: Open Questions

As in the previous chapter, we revisit open questions related to the theory of relaxations and state them explicitly. To our knowledge, most of these problems have not been addressed before and hence we believe that there is ample scope for progress beyond the work presented here.

Rationality of minimum-size relaxations

In Section 7.3.2, we only briefly discussed aspects referring to the rationality of (minimum-size) relaxations. Denoting the smallest number of facets of any *rational* relaxation of a set X by $\mathrm{rc}_{\mathbb{Q}}(X)$, we argued that $\mathrm{rc}(X)$ and $\mathrm{rc}_{\mathbb{Q}}(X)$ differ at most by an additive error of $\dim(X) + 1$ if $X \subseteq \mathbb{Z}^d$ is finite. However, the author is not aware of any finite set X for which $\mathrm{rc}(X)$ and $\mathrm{rc}_{\mathbb{Q}}(X)$ do not coincide.

Problem 10. Does $\mathrm{rc}(X) = \mathrm{rc}_{\mathbb{Q}}(X)$ hold for all finite sets $X \subseteq \mathbb{Z}^d$?

Minimum-size relaxations of the standard simplex

Closely related to the previous problem is the question whether a minimum-size relaxation of a finite set always has to be bounded. As one interesting example, we do not know whether this is the case for $\Delta_d := \{\mathbb{0}, \mathbb{e}_1, \ldots, \mathbb{e}_d\}$, i.e., the set of vertices of the standard simplex. As a consequence, we were not able to give a (provably) tight lower bound on its relaxation complexity. Note that the trivial upper bound is $\mathrm{rc}(\Delta_d) \leq d+1$, while Proposition 8.1.2 contains a much weaker lower bound.

Problem 11. Does $\mathrm{rc}(\Delta_d) = d+1$ hold for every $d \geq 1$?

Computing the relaxation complexity for $d \geq 3$

In Section 7.5.1, we developed an algorithm that computes, for a given (finite) set $X \subseteq \mathbb{Z}^2$, the relaxation complexity of X. Recall that our algorithm was based on the concept of finite "guard sets" that, unfortunately, do not necessarily need to exist in dimension three and higher. One may ask whether in general there are other finite certificates that prove tight bounds on the relaxation complexity. One such certificate might be the following:

Problem 12. Given a finite polyhedral set $X \subseteq \mathbb{Z}^d$, does there always exist a finite set $Y \subseteq \mathbb{Z}^d \setminus X$ such that the smallest number of facets of any polyhedron with $P \cap (X \cup Y) = X$ equals $\mathrm{rc}(X)$?

Note that, given such a finite set Y, it is possible to compute the relaxation complexity of X.

Problem 13. Is there a finite algorithm that, given a set $X \subseteq \mathbb{Z}^d$, computes the relaxation complexity of X?

Asymptotic behavior of $\mathrm{rc}(X)$ for large X

In Section 7.4, we showed that the asymptotic behavior of $\mathrm{rc}(X)$ can be bounded by $O(|X|^{1-1/d})$ if d is fixed. Recall that this was done by proving that the function $g(n,d)$, which has been investigated by Aliev et al. [1], can be bounded by $c^d \cdot n^{1-1/d}$ for sufficiently large n, see Theorem 7.4.3. However, even for $d = 2$ we do not know whether these estimations are asymptotically tight.

Problem 14. For any fixed d, what is the asymptotic behavior of $\mathrm{rc}(X)$ in terms of $|X|$?

Problem 15. For any fixed d, what is the asymptotic behavior of $g(n,d)$?

Higher-degree descriptions

Instead of describing sets as integer solutions of systems of *linear* inequalities, suppose we want to use *polynomial* inequalities. It is an easy exercise to see that for every finite set $X \subseteq \mathbb{R}^d$, there exists one polynomial p such that $X = \{x \in \mathbb{R}^d \mid p(x) \leq 0\}$ holds. However, in general p has large degree and many nonzero coefficients. Thus, given a set $X \subseteq \mathbb{Z}^d$, one may consider the number $\operatorname{rc}_\ell(X)$ defined as the smallest t such hat there exist polynomials $p_1, \dots, p_t$ of degree at most ℓ with

$$X = \{x \in \mathbb{Z}^d \mid p_i(x) \leq 0 \text{ for } i = 1, \dots, t\}.$$

Note that we have $\operatorname{rc}(X) = \operatorname{rc}_1(X)$. There are sets $X \subseteq \{0,1\}^n$ for which there is already a large gap between $\operatorname{rc}_1(X)$ and $\operatorname{rc}_2(X)$. For example, consider the set $X_{\text{even}}(n)$ of binary vectors having an even number of ones. In Section 8.3.3 we argued that $\operatorname{rc}(X_{\text{even}}(n))$ grows exponentially in n. On the other side, as $X_{\text{even}}(n)$ consists of those vectors $x \in \{0,1\}^n$ that satisfy $(k - \sum_{i=1}^n x_i)^2 \geq 1$ for every odd number $k \in \{1, \dots, n\}$, we have that $\operatorname{rc}_2(X_{\text{even}}(n))$ grows only linear in n.

However, using similar arguments as in Section 8.2 one can show that, for fixed ℓ, the value of $\operatorname{rc}_\ell(X)$ for a random set $X \subseteq \{0,1\}^d$ is still exponential in d. Unfortunately, we are not aware of an explicit family of sets having this property.

Problem 16. Fix an integer $\ell \geq 2$. Find an explicit family of sets $X_i \subseteq \{0,1\}^{d_i}$ with strictly growing d_i's such that $\operatorname{rc}_\ell(X_i)$ grows superpolynomially in d_i.

As an interesting candidate, we believe that there is no substantial benefit in using quadratic inequalities instead of linear inequalities in order to describe characteristic vectors of hamiltonian cycles.

Problem 17. Fix an integer $\ell \geq 2$. Does $\operatorname{rc}_\ell(X_{\text{STSP}}(n))$ grow exponentially in n?

Number of additional variables

We have seen several explicit sets $X \subseteq \{0,1\}^d$ whose relaxation complexity grows exponentially in d. On the contrary, in Section 7.1 we argued that all these sets can be described by polynomially many linear inequalities if we allow the use of additional (integer) variables. However, in the mentioned constructions, the number of additional variables also grows (polynomially) in d. One may wonder whether only a "few" additional variables can already help to obtain significantly smaller descriptions as in the case of the original space. Although there are several interesting questions related to this topic, let us focus on a simple particular case. Suppose we are only allowed to use *one* additional variable. That is, given a set $X \subseteq \mathbb{Z}^d$, we search for a system of linear inequalities $Ax + By \leq b$, where B is a matrix having only one column, such that

$$X = \{x \in \mathbb{Z}^d \mid \exists y \in \mathbb{R} : Ax + By \leq b\} \tag{9.1}$$

holds. Applying one round of Fourier-Motzkin elimination, one obtains that the size of such a system has to be at least the square-root of the relaxation complexity of X. In this case, we have no (exponentially) large gap between sizes of relaxations and sizes of descriptions as in (9.1).

Suppose now that we also require y in the description in (9.1) to be integer. Then we can describe the set $X_{\text{even}}(n)$ by only a few linear inequalities as we have

$$X_{\text{even}}(n) = \{x \in \{0,1\}^d \mid \exists y \in \mathbb{Z} : x_1 + \cdots + x_n = 2y\}.$$

Even if we restrict y to only take values in $\{0,1\}$ such a gap is possible: For instance, using hiding sets, one can verify that the relaxation complexity of $X^\star := \{x \in \{0,1\}^n \mid x_1 + \cdots + x_n \neq \lfloor \frac{n}{2} \rfloor\}$ grows exponentially in n. However, it is a simple exercise to describe $X^\star$ by a polynomial-size system of type

$$\{x \in \mathbb{Z}^d \mid \exists y \in \{0,1\} : Ax + By \leq b\}. \tag{9.2}$$

On the other side, we do not believe that a constant number of additional integer variables help to describe sets as $X_{\text{STSP}}(n)$ by polynomially many linear inequalities, but were not able to prove this. We leave a special case of this problem as an open question.

Problem 18. Let $t(n)$ be the smallest number of linear inequalities in a system of type (9.2) for the set $X = X_{\text{STSP}}(n)$. Does $t(n)$ grow exponentially in n?

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